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ABSTRACT

This is part three of a three-part SMSG mathematics text for seventh-grade students. The text was written for those students whose mathematical talent is underdeveloped and is essentially the same subject matter presented in the SMSG text. Chapter topics include: (1) measurement; (2) area and volume; (3) parallels; (4) polygons and prisms; (5) circles; and (6) statistics and graphs. (MP)

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INTRODUCTION TO SECONDARY SCHOOL MATHEMATICS

VOLUME I (Part 3)

(preliminary edition)



SE 087919

INTRODUCTION TO SECONDARY SCHOOL MATHEMATICS

Volume I (Part 3
(preliminary edition)

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Chapter 11

MEASUREMENT

11-1. Counting and Measuring.

Many of the questions people ask in their work or in everyday living begin "How much?" or "How many?" or "How fast?" We ask, "How many people went to the ball game?" or "How much meat shall I buy?" or "How fast can a jet travel?" The answers we want for these questions are all alike in one way: they all use numbers. The answers to some of them are found by counting, while answers to others are found by measuring.

When we ask, "How many people went to the ball game?" we find the answer by counting. Each person is a separate, whole object. There are no fractions of persons at a ball game. They are not all exactly alike, nor are they all the same size, but each is a person. When we count them we use only the counting numbers, and our answer is a counting number. There are exactly 78 or 79 people; it is impossible to have $78\frac{1}{2}$ or $79\frac{1}{4}$ people.

When we ask, "How long is this rope?" we cannot find the answer just by counting. The rope is not made up of separate parts which we can count. We say the rope is one continuous thing. We must find how long it is by measuring. Then the answer is a number of units such as inches or feet. We may say it is 45 inches long, or $3\frac{3}{4}$ feet long, or $1\frac{1}{4}$ yards long. We use a rational number to answer the question, but not always a counting number. Whenever we measure things that are continuous like a rope or a distance, a rational number like $\frac{1}{2}$ yard makes sense. When we count whole items like the runs in a baseball game, a fraction of a score has no meaning.

To answer the question, "How fast can a jet travel?", we must use two units. We say, "It can travel 620 miles per hour", using a unit of distance (miles), and a unit of time, (hour). Is distance a continuous quantity? Does a fraction of an hour have meaning? Is time a continuous quantity? In this chapter we shall study ways to measure quantities that are continuous, instead of

things which can be counted. The word "quantity" is used to refer to anything that has size.

Exercises 11-1

Which of the following questions (1-6) would you answer by counting? by measuring?

1. How many people are in this class?
2. How far is it to the moon?
3. How big a crowd went to the picnic?
4. How cold is it today?
5. How old are you?
6. How much water is there in the swimming pool?

Which of the following are continuous quantities?

7. Your height.
8. Your weight.
9. The size of your family.
10. Your collection of pens.
11. The length of this page.
12. The odd numbers between 0 and 10.
13. The amount of air in this room.

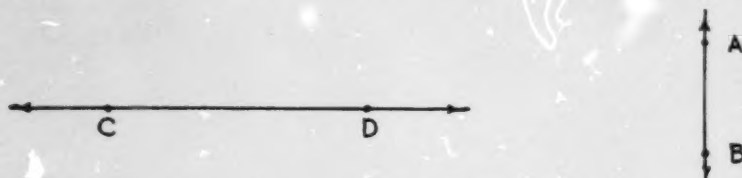
11-2. Properties of Continuous Quantities.

In answering the question "How far?", we usually measure a line segment. A segment like a stretched wire is an example of a continuous quantity. The interior of a closed curve is another example of a continuous quantity. The ceiling of a room is an example of the interior of a closed curve. Many continuous quantities which we wish to measure may be thought of as line segments or interiors of simple closed curves. A group of 12 boys is an example of a quantity that is not continuous.

Discussion questions:

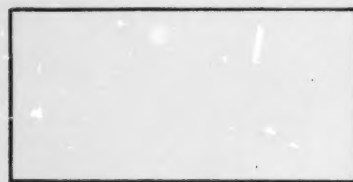
1. What are five examples of continuous quantities?
2. What are five examples of quantities which are not continuous?

One way to talk about the size of continuous things is to compare them. To find out which of two boys is taller, the boys might stand back-to-back to compare the positions of the tops of their heads. A similar method can be used to find out which segment $\overline{AB}$ or $\overline{CD}$, shown below, is longer. The symbol $\overline{AB}$ was used in Chapter 4 as a short way of writing "the segment AB".



To compare these segments, lay the edge of a piece of paper along $\overline{CD}$ and mark points C and D on this paper so that they match the points in the picture. Place the edge of the paper along $\overline{AB}$ with point C on point A. Where does point D fall? If it is between A and B, $\overline{AB}$ is longer than $\overline{CD}$. If D falls on B, the segments are the same length. If B is between C and D, $\overline{CD}$ is longer than $\overline{AB}$.

Look at the simple closed curves below. Copy them by tracing them on a sheet of thin paper.



A



B

To compare the size of the interior of curve A with the size of the interior of curve B, cut out a copy of figure B and place it in the interior of curve A. Figure B separates the interior of A into the parts covered by the interior of B, and the part not covered. We say that the size of the interior of curve A is larger than the size of the interior of curve B.

When we use methods like this to compare the sizes of line

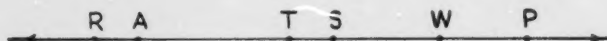
segments or of interiors of simple closed curves, we assume (take for granted) these properties of geometric quantities that are continuous.

Motion Property: A geometric figure may be moved without changing its size or shape.

Comparison Property: Two continuous quantities of the same kind may be compared to find out whether they have the same size, or which one is the larger.

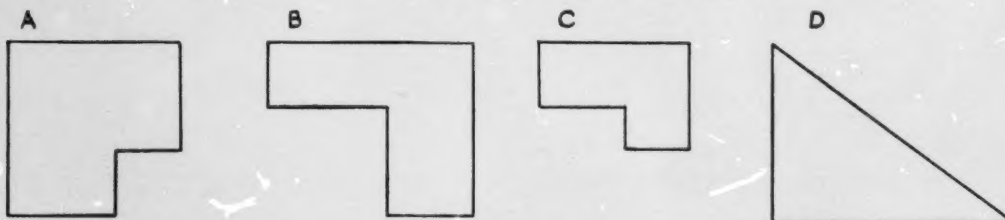
Exercises 11-2a

1. Cut a strip of paper or string of the same length as $\overline{RT}$, a segment of the following line:



Use it to find whether the following line segments have same length or which is longer.

- a. $\overline{RT}$ and $\overline{SP}$.
 - b. $\overline{RT}$ and $\overline{TW}$.
 - c. $\overline{RT}$ and $\overline{AS}$.
 - d. $\overline{RT}$ and $\overline{AT}$.
 - e. $\overline{RT}$ and $\overline{TS}$.
2. a. Copy the figures below and cut out the one you think has the smallest interior. Place it on Figure A, and shade the part of the interior of A which is not covered by the smaller figure. In the same way compare the smallest with each of the other figures.

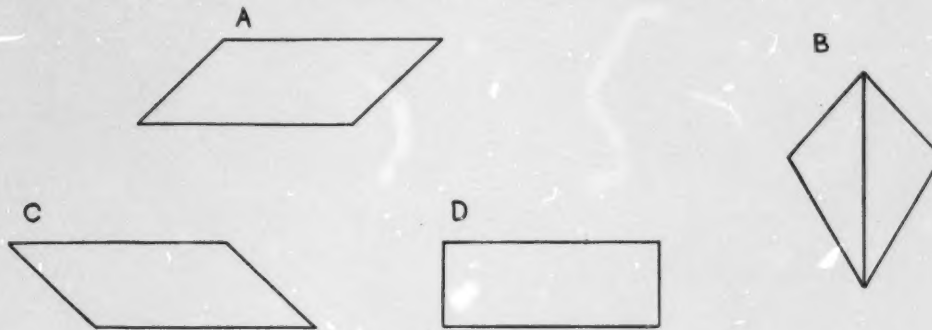


- b. Using one of the symbols, $>$, $=$, or $<$, write sentences comparing the size of the interior of the smallest figure

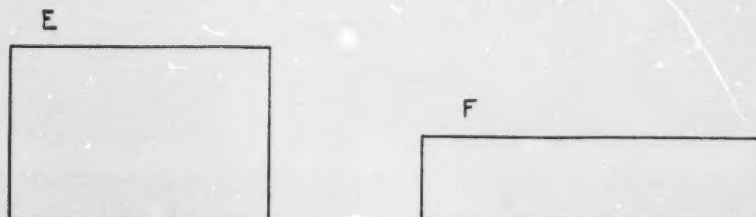
with the size of the interior of each of the other figures, like this:

The interior of figure ? > the interior of the figure ? .

3. a. Which of the four figures below do you think has the largest interior? List them in what you think is the order of their sizes, putting the largest one first.



- b. Cut out three copies of Figure A, and use them to compare the interior of Figure A with the interior of each of the others. (Cut up figure A if necessary.)
- c. Was the list you made for Problem 3a correct?
4. a. Cut a strip of paper or a piece of string and mark off on it the lengths of the four line segments in the closed curve in Figure E below. Then use the strip to compare the length of the closed curve in Figure E with the length of the closed curve in Figure F. Which curve has the greater length?
- b. Cut out a copy of Figure E and use it to compare the sizes of the interiors of Figure E and Figure F. Which interior has the larger area?



5. Draw a square and label the vertices ABCD. Cut out your square and then cut along line segment $\overline{AC}$. Put the two parts together in some other way so that they do not form a square, and trace around the boundary of the new figure. How does the interior of your new closed curve compare in area with the interior of your square ABCD? How does the length of the boundary of the new closed curve compare with the boundary of the square?



In Problems 2-5 we have assumed two more properties of geometric continuous quantities.

Matching Property: Suppose two geometric continuous quantities are both made up of parts. Suppose that every part of one can be matched to a part of the same size in the other. Then the two continuous quantities have the same size.

Subdivision Property: A geometric continuous quantity may be subdivided in any way we want.

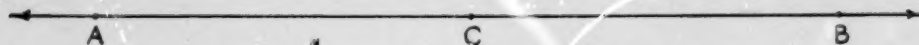
The subdivision property means simply that a quantity may be separated into parts. When a farmer plans to plant wheat and oats on his land he must first "subdivide" or separate the land into parts to be planted to each crop. Perhaps some of you live in an area called a "subdivision." It was made into building lots by subdividing (or separating into parts) a larger piece of land. In neither of the above examples was any of the original land lost, nor any new land gained.

Subdividing Continuous Quantities.

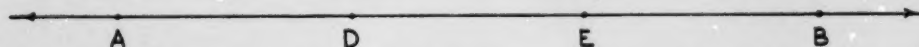
Lines and the interiors of closed curves can be subdivided. The subdivision property of a continuous quantity is important

because it is the basis for measuring. Some examples will help you understand this idea.

Each of the next five figures is a picture of the same segment, $\overline{AB}$. Point C subdivides segment $\overline{AB}$ into two segments $\overline{AC}$ and $\overline{CB}$. How do segments $\overline{AC}$ and $\overline{CB}$ compare in size?



In the next sketch, $\overline{AB}$ is subdivided by two points D and E, so that $\overline{AD}$, $\overline{DE}$, and $\overline{EB}$ are equal in length.

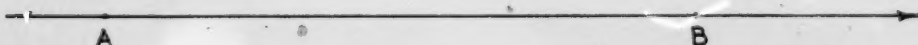


How does the length of $\overline{AD}$ compare with the length of $\overline{AB}$?

How does the length of $\overline{AE}$ compare with the length of $\overline{AB}$?
Of $\overline{DB}$? Of $\overline{DE}$?

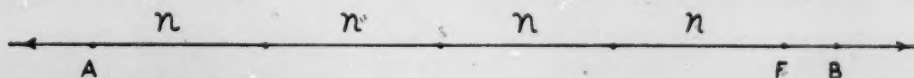
$\overline{AB}$ can be subdivided in other ways to compare the length of one line segment with the length of another line segment. Choose a segment of any length less than the length of $\overline{AB}$ and call the length of the segment n .

n

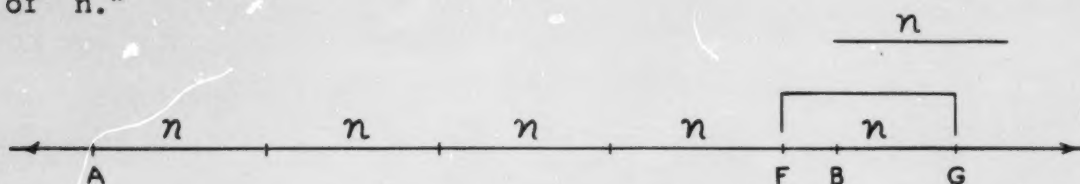


Use a strip of paper or a compass to mark off on $\overline{AB}$ segments of length n . Begin at point A. In the figure below, a segment of length n is marked off 4 times so that $\overline{AF}$ is as long as $4n$. The symbol " $4n$ " means "4 times as long as n ." Point F is between A and B.

n

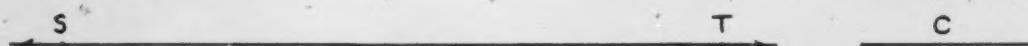


If a fifth segment n is marked off, segment $\overline{AG}$ is as long as $5n$, and B is between A and G . So $\overline{AB}$ is longer than $4n$ but shorter than $5n$. Since $\overline{BG}$ is longer than $\overline{FB}$, we say that $\overline{AB}$ is approximately (or "just about") equal to $4n$. There is a symbol for the words "approximately equal to"; It is a wavy equal sign like this, " $\approx$ ". We could use this symbol and state the length of $\overline{AB} \approx 4n$. This may be read "The length of segment $\overline{AB}$ is approximately equal to 4 times the length of n ."

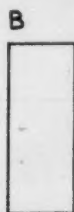


Exercises 11-2b

- Copy the line segment $\overline{ST}$ and the line segment marked " c " below. Use a strip of paper the length of c (or use a pair of compasses) to compare the length of line segment $\overline{AB}$ with the segment of length c . Show your subdivisions as in the sketch of line segment $\overline{AB}$ above, and state your result.

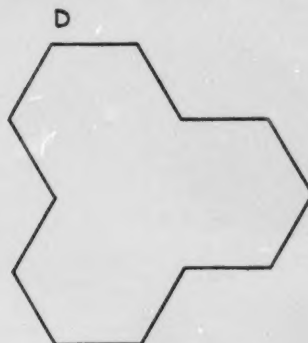
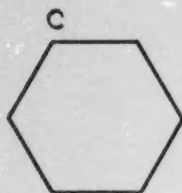


- Use a similar method to compare the interiors of two closed curves. Copy curves A and B below. Cut out Figure B and subdivide the interior of Figure A into parts the same size as the interior of Figure B.



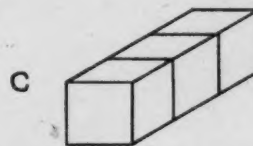
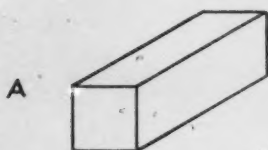
- The size of the interior of curve A is approximately ? times the size of the interior of curve B.

3. a. Compare the length of curve D with the length of curve C.

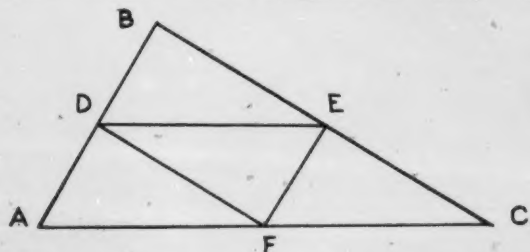


- b. Compare the size of the interior of curve D with the size of the interior of curve C.

4. Look at the box A in the figure. We could think of the space in the interior region as subdivided into parts the size of box B, as shown in C. The interior of box A is about ? times as large as the interior of B.



5. Draw a triangle ABC. On each side of the triangle place a point which subdivides the side into line segments with the same length. Name these points D, E, and F, as shown in the sketch. Draw line segments $\overline{DE}$, $\overline{EF}$, and $\overline{DF}$.



- a. Use a strip of paper to determine which line segments have the same length. (You should find three sets, with three in each set.)

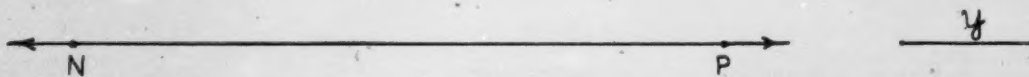
- b. Name all the simple closed curves you see in the figure.
(Try to find eleven curves.)
- c. Copy triangle EFC by tracing it. Cut it out and compare its interior with the interiors of the other triangles.
- *d. What sets of simple closed curves have interiors of the same size? (One set should have four members, and two sets should each have three members.)

11-3. Measuring.

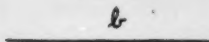
In Section 2 the length of segment $\overline{AB}$ was compared with the length of another segment called n . The comparison was made by subdividing $\overline{AB}$ into parts, each of which had the length n . It was found that the length of $\overline{AB} \approx 4n$. This process is called measuring. The segment of length n is a unit of measure. The number 4 is the measure of $\overline{AB}$ when it is measured in unit n .

Class Exercises 11-3a

1. Copy line segment $\overline{NP}$ below. Divide line segment $\overline{NP}$ into subdivisions as long as segment y . $\overline{NP}$ is approximately ? times as long as y .



2. Copy line segment $\overline{NP}$ again, and subdivide it to compare its length with a segment of length b . $\overline{NP}$ is about ? times as long as b .



You should see that if you measure a segment with units of different size, the measure above does not tell how large the segment is. The unit of measure that is used must be stated also. The length of a segment includes both the measure and the

unit of measure. When y is used as a unit of measure, the length of $\overline{NP} \approx 5y$.

Notice how these words are used:

5 is the measure
 y is the unit
 $5y$ is the length

Exercises 11-3b

1. a. What unit of measure did you use in Problem 1 of Exercises 11-2b?
b. What was the measure of $\overline{ST}$?
2. a. What unit of measure did you use in Problem 2 of Exercises 11-2b?
b. What was the measure of the interior of curve A?
3. a. In Problem 3a of Exercises 11-2b, suppose you use one segment in curve C as a unit of measure, and call the length of this segment "t." Then what is the length of curve C?
b. Using the same unit of measure "t," how long is curve D?
c. How does the length of curve D compare with the length of curve C?
d. In Problem 3b, what unit of measure did you use? What is the size of the interior of curve D?
4. In Problem 4, what was used as the unit of measure to measure the interior of Box A? How large is the interior of Box A?

You have seen that the sizes of things which are continuous can be found by measuring. To do this, you must use a unit of measure. Two things must be true of a unit of measure if it is to be satisfactory for geometric quantities.

1. It must be of the same nature as the thing to be measured. A line segment must be used to measure a line segment; the interior of a simple closed curve must be used to measure the interior of a simple closed curve, and so on.

2. It must be possible to move the unit, or to copy it exactly. It can then be used to subdivide the thing that is measured.

Often people use as a unit of measure anything which is conveniently at hand. Parts of your body like fingers, arms, or feet are always with you. They are often used as units of measure. Why are these units of length not completely satisfactory for us to use?

Exercises 11-3c

1. Use the length of the middle section of your middle finger as a unit of measure, and measure (a) the length of your desk; (b) the length of this page.
2. Use the length of your foot to measure the length of your classroom.
3. Compare your answers to Problems 1 and 2 with the answers of your classmates. How do you explain any differences?
4. When you used your finger as a measuring instrument, what characteristics of your finger did you have to ignore? What did you have to ignore when you used your foot as a unit of measure?
5. What object would make a convenient unit to measure the space in the interior of a desk drawer?
6. Can you think of some object that can be moved and that could be used as a unit of measure for the surface of your desk? Could you use a sheet of paper from your notebook? Place several sheets side by side, and count the number of notebook-sheet units on the surface of your desk. What is the measure of the surface? How large is the surface?

7. Did you and all of your classmates use the same sized notebook-sheet unit? If not, what was true of the measure?
-

11-4. Standard Units of Length.

Units of measure like the width of your hand or the length of a strip of paper are not always the same. Your hand or strip of paper may be of different size from those of your friends. In order to have confidence in sizes of measurements, we need to have units of measure that are the same for everybody. Usually the laws of a country state the units of measure to be used. When a unit of measure is established so that it is always the same size, it is called a standard unit. When you buy a ruler, the marks on it represent line segments that are units of measure of length. Your friend's ruler has the same size units. This is so simple that it may seem strange that it took a long, long time and much hard work to be able to make these units the same.

People of caveman times had little need for a measuring unit that was the same as everyone else's. Without machinery or buildings, man had little need to measure. The few things he owned could be traded with little concern for size. As civilization developed there were more and more goods to be traded, buildings to be built, and finally machines to be made. These new activities made sizes of goods and sizes of parts very important. People now have great need to measure with units that are the same size for everybody.

Linear Measure.

The measurement of lines is called linear measurement. When we travel between any two points, linear measures tell us the distance we have traveled. The first units used for linear distance were parts of the body like the foot or hand. A man's foot is about as long as the unit foot. The distance from the tip of the nose to the end of the fingers when the arm is held straight out is about as long as a yard. Horses are still measured by a unit called a hand. A hand is the width of the

average hand and is about the same as 4 inches. When boys play marbles, they sometimes measure with a unit called the span.

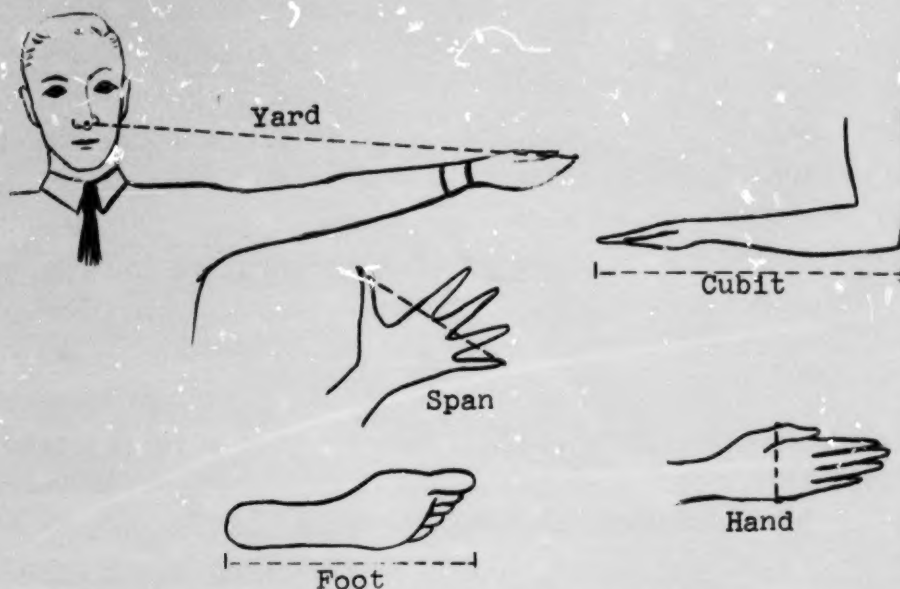


Figure 11-4a

A span is the distance between the tips of the thumb and little finger when they are spread apart as far as possible. When Noah built the ark, it was supposed to be 300 cubits long, 50 cubits wide and 30 cubits high. What is the length of a cubit? Sailors measure water depth in fathoms. How long is a fathom? When the Roman soldiers marched, they counted their paces. The Roman pace, however, was a double step. The Latin words for "a thousand paces" are "milia passuum". Our "mile" comes from "milia" which means 1000. What is the length of your pace?

Units of measure like the pace or hand are not all the same. Different people have hands of different size. When people began to trade and travel, it became important to have units of measure that are always the same. Gradually the sizes of the "foot", "yard", and other units, were described more precisely. Thus they became "standardized." Finally a group of scientists developed an orderly, complete system of measurement called the metric system. The meter is the basic unit of length in this system. One meter is a little longer than a yard. The length of a meter

was chosen to be one-ten millionth (.0000001) of the distance from the North Pole to the Equator.

The National Bureau of Standards in Washington has an accurate copy of the meter. This bar is made of platinum and iridium, a metal which changes very little in length when temperature and air pressure change. Congress has passed a law that tells the fraction of this bar which is the official yard in the United States. This bar is the standard unit of length with which all standard units of measurement in this country are compared. The bar is considered so important that it is locked up very securely. Recently the meter has been defined more precisely in terms of the length of light waves.

Exercises 11-4a

1. Homework exercise.

Find at home and bring to class labels to show the standard unit in which the following products are measured. If a label cannot be found, try to find some other way of learning about the units.

- | | |
|-----------------|----------------|
| a. bread loaves | f. cake tins |
| b. canned foods | g. light bulbs |
| c. men's hats | h. bicycle |
| d. women's hats | i. movie film |
| e. socks | |

2. Use a piece of tagboard or cardboard with a straight edge and make a 6-inch ruler. Mark one edge with 1-inch segments and the opposite edge with $\frac{1}{2}$ -inch segments.
3. Make two more 6-inch rulers, one marked with $\frac{1}{4}$ -inch intervals and the other with $\frac{1}{8}$ -inch spaces.
4. Measure each of these lines to the nearest half inch using the ruler that you made in Problem 2. Use the symbol $\approx$ in your answers to Problems 4-8 to show that your measurements are approximate.

- a. _____
b. _____
c. _____
d. _____
e. _____
5. Measure each of the above segments to the nearest $\frac{1}{4}$ inch, using your $\frac{1}{4}$ -inch ruler.
6. Measure each of the above segments to the nearest $\frac{1}{8}$ inch, using your $\frac{1}{8}$ -inch ruler.
7. Measure each of the above segments using the middle section of your third finger.
8. Which measuring unit gave you the most satisfactory results? Why?
- _____

Measuring line segments is like comparing the line segment with a number line. The 12-inch ruler you use is a part of a number line in which the unit is an inch. The part shown on the ruler starts at zero and goes to 12. It is marked to show counting numbers and some of the other rational numbers. When you measure a segment, you compare the length of the segment with the number line marked on the ruler.

Class Exercises 11-4b

Let's stop and take a careful look at the construction of a ruler.

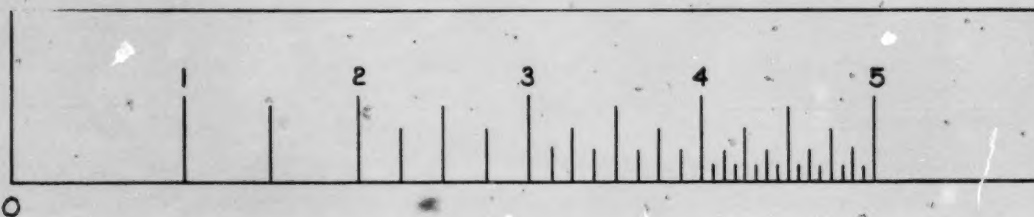


Figure 11-4b

The ruler is divided into 12 equal units that everyone knows as inches. The space at each end of the picture on the preceding page is an inch long.

1. Why is it necessary to divide the units?
2. How is the inch between 1 and 2 divided?
3. How are the other inches divided?
4. When divisions smaller than those shown are needed, how are they obtained?
5. How many divisions are there in the third inch? The fourth inch? The fifth inch?
6. How many divisions is it possible to place between any two of the smallest divisions on your ruler?

One way to use a ruler to measure the length of $\overline{AB}$ is shown in the figure below.

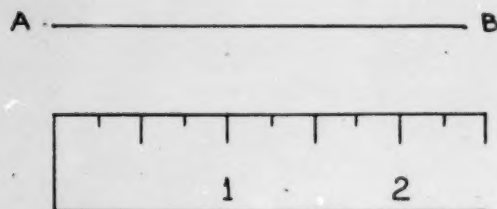


Figure 11-4c

Place the zero point of the ruler at the left end of the segment. The length of the segment is the number of the point on the ruler that matches the right end of the segment. What is the length of $\overline{AB}$?

Suppose your ruler were broken near the 2-inch mark. Can you still use it to measure segments? The diagram below shows a broken ruler placed to measure segment $\overline{CD}$.

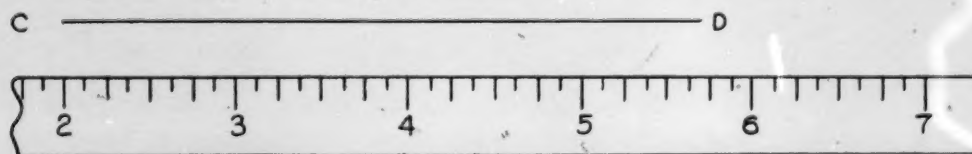
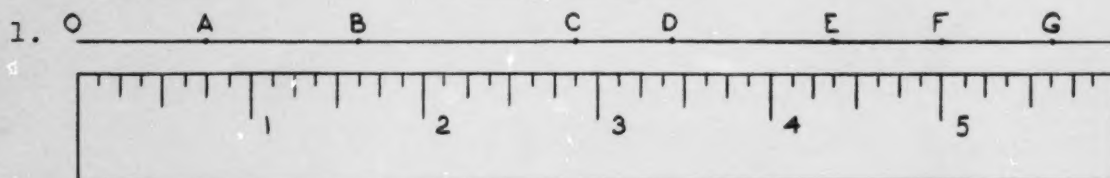


Figure 11-4d

How long is $\overline{CD}$, to the nearest $\frac{1}{8}$ inch? Is it $5\frac{3}{4}$ inches?

When you place a ruler so that the end of a segment lies on some other mark than zero, you must be careful to make a correction in the length that is read on the ruler. In the diagram on the preceding page the length of the segment is $5\frac{3}{4}$ in. - 2 in., or $3\frac{3}{4}$ in. It is sometimes easier to measure a segment accurately if this method is used, as long as you remember to make the necessary correction.

Exercises 11-4c



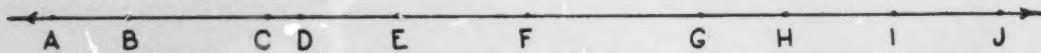
What point on the ruler is directly below each of the points, A through G, on the line segment?

2. How long is:
 - a. $\overline{AB}$? b. $\overline{AC}$? c. $\overline{CD}$? d. $\overline{DE}$? e. $\overline{FG}$?
3. a. Draw a segment 5 inches long and divide it into sections each $\frac{5}{8}$ inch long.
 - b. Divide 5 by $\frac{5}{8}$.
 - c. Is there any relation between parts (a) and (b) of this problem?
4. a. Draw a line segment across your paper and mark these segments on it so that the left end of one starts where the right end of the preceding segment falls:

$2\frac{1}{2}$ "; $1\frac{1}{4}$ "; $\frac{5}{8}$ "; $\frac{5}{16}$ ".

 - b. What is the total length of these segments? Read this answer on your ruler.
 - c. Check your answer by addition of fractions.

5. Measure with your ruler the segments marked on the line below as indicated:



- | | |
|--------------------|--------------------|
| a. $\overline{AB}$ | e. $\overline{EF}$ |
| b. $\overline{BE}$ | f. $\overline{EJ}$ |
| c. $\overline{AJ}$ | g. $\overline{GH}$ |
| d. $\overline{CG}$ | h. $\overline{IJ}$ |
6. a. On the preceding line, measure each of the following segments if you have not already done so: $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, $\overline{HI}$, and $\overline{IJ}$.
- b. Add all of these measures and check with the answer in 5(c) to see if they agree.
7. Write the fractions $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{8}$, and $\frac{1}{16}$ as common fractions in base two numerals.
- *8. Is there any relation between the division on a ruler and base 2?

There are many standard units of length, such as the inch, foot, yard, or mile. Just as there are many names for the same rational number like $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{10}{15}$, so there are many names for the same length, such as 2 feet, 24 inches, and $\frac{2}{3}$ yard. It is important to be able to change the name of a measurement from one form to another. To change these names, you must know the relationships among the units. It is also necessary to understand the meaning of the symbol " $=$ ". Up to this time this symbol has been used to show different names for the same number. As it is used here, " $=$ " means different names for the same length. When you work with measures, you use this meaning for the equal sign. Of course, the measure could be a measure of area, of temperature, of time, or of any other continuous quantity besides length.

As an example, let us do some problems in changing inches to feet and feet to inches. We know the basic fact that 12 in. = 1 ft. This means that 1 in. = $\frac{1}{12}$ ft. also. With this information let us change 60 inches to feet. We can write the problem as a number sentence, thus: 60 in. = ? ft. From the meaning of measure, 60 inches is 60 $\cdot$ 1 in. The number sentence can then be rewritten as 60 $\cdot$ 1 in. = ? ft.

Since 1 in. and $\frac{1}{12}$ ft. are different names for the same length, we can replace 1 inch by $\frac{1}{12}$ ft. Then 60 $\cdot$ $\frac{1}{12}$ ft. = ? ft.

Now the arithmetic on the left side of the number sentence can be done and we see that 60 in. = 5 ft.

The same thinking is used when 60 ft. is changed to inches. The basic fact (1 ft. = 12 in.) is the same, and the problem looks like this:

$$\begin{aligned} 60 \text{ ft.} &= \underline{\quad ? \quad} \text{ in.} \\ 60 \cdot 1 \text{ ft.} &= \underline{\quad ? \quad} \text{ in.} \quad \text{Since 60 ft. means } 60 \cdot 1 \text{ ft.} \\ 60 \cdot 12 \text{ in.} &= \underline{\quad ? \quad} \text{ in.} \quad \text{Replacing 1 ft. by 12 in.} \\ &= \underline{720} \text{ in.} \quad \text{Completing the arithmetic.} \end{aligned}$$

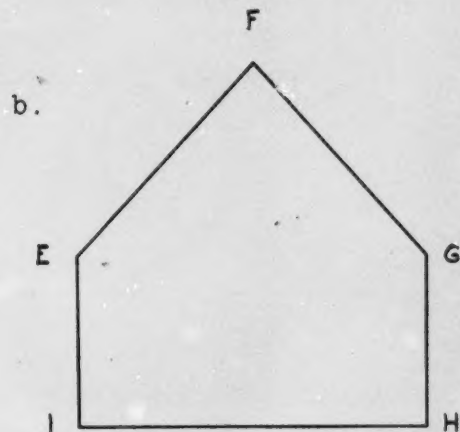
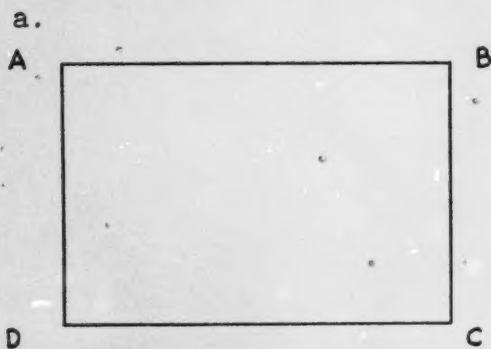
Common sense also helps in checking to see whether answers are sensible. If the units of a measurement are small, it will take more of them to show a measurement than when the units are large. Always check your answers to see that they are sensible.

Most of the basic relationships needed to change the name of units are familiar to you. They are given in the tables at the end of Chapter 12. The table of common linear measures is also given here.

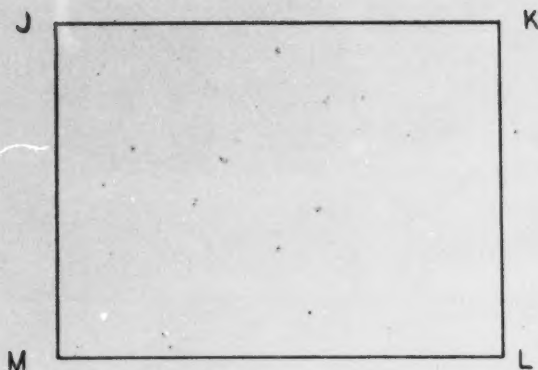
$$\begin{aligned} 12 \text{ in.} &= 1 \text{ ft.} \\ 36 \text{ in.} &= 1 \text{ yd.} \\ 3 \text{ ft.} &= 1 \text{ yd.} \\ 5280 \text{ ft.} &= 1 \text{ mi.} \end{aligned}$$

Exercises 11-4d

1. Use the method described above to change the following lengths to different units.
 - a. 4 ft. = _____ in.
 - b. 27 ft. = _____ yd.
 - c. 27 yd. = _____ ft.
 - d. 100 yd. = _____ ft.
 - e. 120 ft. = _____ yd.
 - f. 72 in. = _____ ft.
 - g. 4 mi. = _____ ft.
 - h. 54 in. = _____ ft.
 - i. 60 in. = _____ yd.
 - j. 27 in. = _____ yd.
2. How many yards are there in one mile?
3. Walk naturally for ten steps. Measure the distance from the first toe mark to the last toe mark. Divide by ten to find the length of your pace. Express this length in two ways:
 - a. in feet,
 - b. in inches.
4.
 - a. Use your pace to measure the length and width of the gym in your school. Change your answer to feet.
 - b. Measure the length and width of the gym with a yardstick. Measure to the nearest foot and check with your answer to part (a).
5. Find the total length of each of the following simple closed curves by measuring each segment and adding all the measures together. Use a ruler marked in 16ths of an inch.



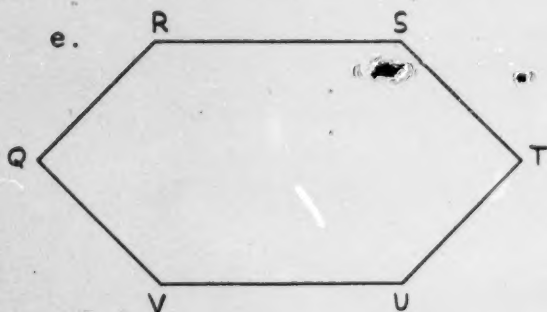
c.



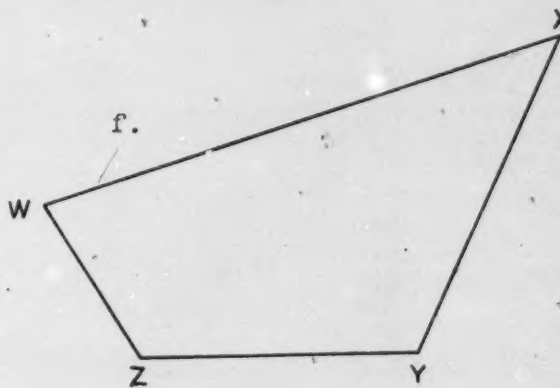
d.



e.



f.



6. a. Use 18 inches as the length of a cubit and find the dimensions (length, width, and height) of Noah's Ark. You can find the dimensions in cubits near the beginning of Section 11-4.
- b. Give the dimensions of the Ark in feet and also in yards.
-

11-5. Precision of Measurement.Class Exercises 11-5a

1. Construct a table like the one shown below. Use the table for recording results of the exercises that follow.

Segment	Measures, to the nearest			
	inch	$\frac{1}{2}$ inch	$\frac{1}{4}$ inch	$\frac{1}{8}$ inch
$\overline{AB}$	3	$3\frac{1}{2}$		
$\overline{CD}$				
$\overline{EF}$				
$\overline{GH}$				
$\overline{IJ}$				

2. Measure each of the segments below using the ruler you made with only inch marks. You will be measuring to the nearest inch. Write your measures in the first column of the table. For example, $\overline{AB}$ is nearer 3 inches than 4 inches, so "3" is written opposite $\overline{AB}$ in the table.

A _____ B

C _____ D

E _____ F

G _____ H

I _____ J

3. Use the ruler with only $\frac{1}{2}$ -inch marks and measure each segment. You will be measuring to the nearest half-inch. Record the measures in the table.
4. Measure each segment with your $\frac{1}{4}$ -inch ruler and with your $\frac{1}{8}$ -inch ruler, and record the measures.

5. a. Which measurement for segment $\overline{AB}$ is nearest the true length of the segment?
 b. Which measurement is nearest the true length of $\overline{EF}$?
 c. Why is this true?
6. If we asked the same questions (5) for each of the other segments, would you give the same answers? If not, you will see the reason a little later.

When a small unit of measure, or subdivision, is used we say the measurement is more precise than when a larger unit is used. A measurement expressed to the nearest $\frac{1}{4}$ inch is more precise than one expressed to the nearest $\frac{1}{2}$ inch. A measurement made to the nearest inch is more precise than one to the nearest foot. The precision of a measurement depends on the size of the smallest subdivision used in measuring. The measures in the last column of the table were obtained by using a $\frac{1}{8}$ inch unit. We say that those measurements have a precision of $\frac{1}{8}$ inch.

7. How could you get a more precise measurement for the above segments than any you have in the table?
8. What is the size of the very smallest subdivision of an inch that could be used in measuring?

Suppose you wanted to measure the thickness of a hair. Would you use a ruler marked in inches? in $\frac{1}{8}$ inches? in $\frac{1}{16}$ inches? You probably said that these measuring tools were too "coarse," that you needed a "finer" instrument. Let us say that we need a more precise instrument. Do you know what a "precision instrument" is? The term applies to a wide variety of tools and instruments that are made with such great care that "fine" measurements can be made. This care is necessary if the instrument is to be used for very precise measures.

If very precise measures are needed to make a machine run properly, that machine is also called a precision instrument. A watch is one precision instrument. The family auto is another; it would not work very well if one of the cylinders were $\frac{1}{16}$ of an inch too large and likewise, the piston would not even go into the cylinder if the cylinder were that much too small. Machinists sometimes must take measures correct to the nearest thousandth or ten-thousandth of an inch. This is such a small measurement that your eye cannot see it, but there are instruments that can. One such precision instrument is the micrometer pictured at the end of this paragraph. With it measurement can be made to the nearest .001 of an inch. Perhaps someone in the class can bring one to school and demonstrate how it works.

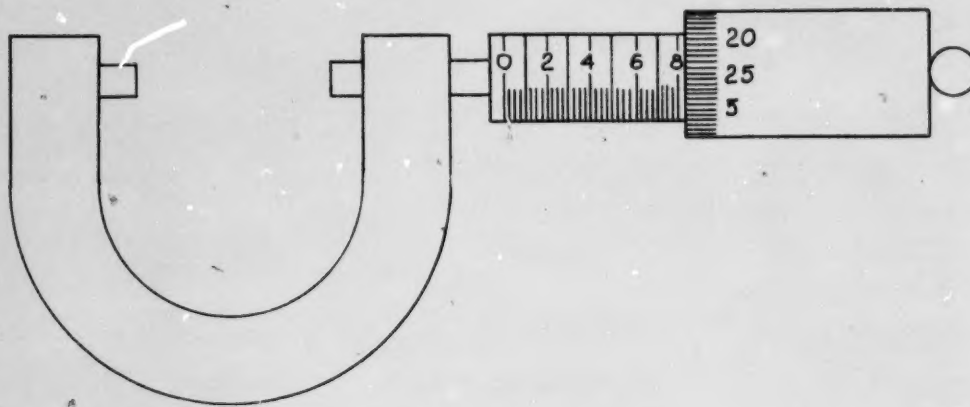


Figure 11-5

Exercises 11-5b

1. Which of these fractions is smaller?

a. $\frac{1}{8}$ or $\frac{1}{4}$

e. $\frac{1}{100}$ or $\frac{1}{10}$

b. $\frac{1}{2}$ or $\frac{1}{4}$

f. $\frac{1}{5}$ or $\frac{1}{2}$

c. $\frac{1}{8}$ or $\frac{1}{16}$

g. $\frac{1}{1000}$ or $\frac{1}{100}$

d. $\frac{1}{8}$ or $\frac{1}{10}$

h. $\frac{1}{2}$ or $\frac{1}{10}$

2. Which of the two measures is more precise?

- | | |
|---|---|
| a. 8 in. or $7\frac{1}{2}$ in. | e. 4.6 mi. or 10 mi. |
| b. $3\frac{1}{4}$ ft. or 2 ft. | f. 2 ft. or 24 in. |
| c. 6 in. or $5\frac{3}{4}$ in. | g. $5\frac{12}{16}$ in. or $5\frac{3}{4}$ in. |
| d. $3\frac{3}{4}$ in. or $3\frac{7}{8}$ in. | |

Class Exercises 11-5c

You can make the table you constructed for Exercises 11-5a do double duty by using it for this set of exercises.

1. a. Cover the column headings in your table. Do the numerals in the first row tell you the size of the subdivisions of an inch that were used in measuring $\overline{AB}$? They do if you have 3, $3\frac{1}{2}$, $3\frac{1}{4}$, $3\frac{3}{8}$, because "3" shows whole inches, " $3\frac{1}{2}$ " shows half inches, and so on.
- b. Do the measures for $\overline{CD}$ show the size of the subdivisions that were used? If the last two columns show $1\frac{3}{4}$, $1\frac{3}{4}$, the answer is "no." These numerals both show $\frac{1}{4}$'s, and the last one was obtained by using $\frac{1}{8}$'s. Since $\frac{3}{4}$ can be written as $\frac{6}{8}$, the last measure should be $1\frac{6}{8}$ to show the precision that was used.
2. Check the measures in the last two columns for the rest of the segments, and correct them if necessary to show the precision of measurement.
3. Look at the first two measures for $\overline{CD}$. Do they both read "2"? Then does the second numeral show that you measured with $\frac{1}{2}$ -inch precision? Can you find a way of correcting this measure?
4. Now check all the measures in the table to see that the precision of all the measures is shown.

The important ideas contained in Section 11-5 are listed below.

1. The precision of a measurement depends on the size of the smallest subdivision used in measuring. For example, a measurement made with $\frac{1}{8}$ -inch subdivisions is more precise than one made with $\frac{1}{4}$ -inch subdivisions. Of course, anyone can make a mistake in reading the marks on a ruler or be careless in placing it along the segment to be measured. This is not what is meant by precision of measurement.
2. A measurement should be stated in a form that shows its precision. For example, the measurement of a segment made with precision of $\frac{1}{8}$ inch should be written " $6\frac{6}{8}$ in." instead of " $6\frac{3}{4}$ in," or " $4\frac{0}{8}$ in." instead of "4 in."

Exercises 11-5d

1. Suppose each of the following measurements is correctly stated. What was the size of the unit or subdivision used in measuring each?

a. 3 ft.	e. $4\frac{1}{2}$ ft.
b. 9 in.	f. 20 mi.
c. $2\frac{3}{4}$ in.	g. $5\frac{12}{16}$ in.
d. 8.7 mi.	h. $9\frac{7}{8}$ in.
2. Use your commercial ruler to measure the lengths of $\overline{KL}$ and $\overline{MN}$ with precision of

a. $\frac{1}{2}$ inch	b. $\frac{1}{4}$ inch	c. $\frac{1}{8}$ inch	d. $\frac{1}{16}$ inch.
-----------------------	-----------------------	-----------------------	-------------------------

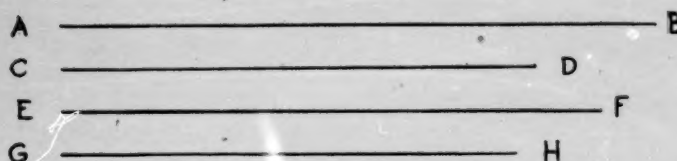
Write the measurements to show the precision in each case.

K _____ L

M _____ N

11-6. Greatest Possible Error.

Measure each of the segments below with 1-inch precision.



You probably have the same result, 3 inches, for each segment. Does this mean:

- (1) That all the segments are the same length?
- (2) That you made a mistake in measuring?
- (3) That each of the segments is more than 3 in.?
- (4) That each of the segments is less than 3 in.?
- (5) That there is an error in the 3-inch measurements?

You probably answered "No" to each of the first four questions. Two of the segments were less than 3 inches in length; two were more than 3 inches. You made no mistake in the measurements. The length of each segment was nearer 3 inches than 2 inches or 4 inches. Therefore, to the nearest inch these segments are each 3 inches long. The right end of a 3-inch segment might fall anywhere between the arrow points shown on the ruler pictured below.

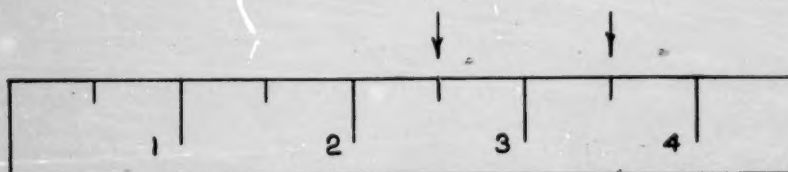


Figure 11-6a

How did you answer Question (5)? Is there an error involved in the 3-inch measurements? Yes, because none of the segments is exactly 3 inches. The error is made because a measurement tells only whether a segment is nearer one length than another. We can never measure exactly, no matter how small a subdivision is used.

Let us see how much error is made when we measure with different degrees of precision.

Figure 11-6b shows the possible lengths of some segments which might be called 3 inches.

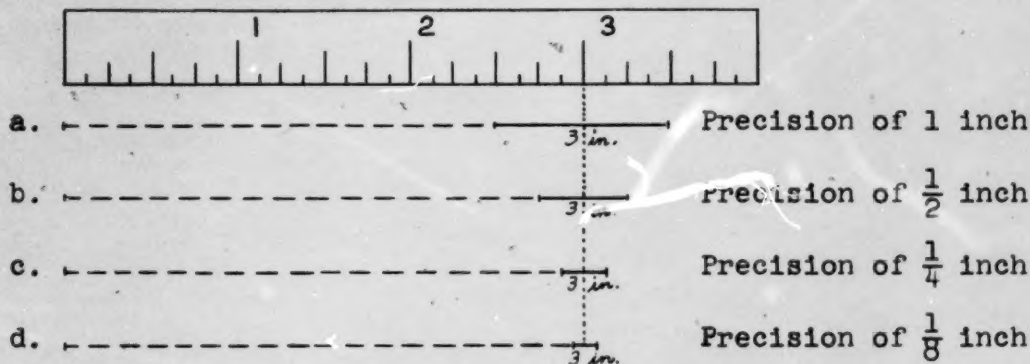


Figure 11-6b

If segment a is measured with precision of 1 inch, the end-point of the segment might fall anywhere within the solid part of the segment. This solid part is 1 inch long. If segment b is measured with precision of $\frac{1}{2}$ -inch, the end-point must fall within the solid part of the segment b ($\frac{1}{2}$ inch). Study of the diagram shows that the precision of a measure tells the limits between which the end-point of a segment must lie.

Class Exercises 11-6a

1. Draw the longest possible segment that would be measured as 3 inches if you were measuring to the nearest inch. Draw the shortest possible segment.
2. What could be the true length of the longest segment? the shortest?
3. What is the greatest possible difference between the true lengths of the two segments and 3 inches?
4. Draw the longest and the shortest segments that you would measure as $2\frac{1}{2}$ inches, if you were using only half-inch subdivisions.

5. What could be the true length of your longest segment? the shortest?
6. What could be the greatest possible difference between the real lengths of the segments described in Problem 4, and $2\frac{1}{2}$ inches?
7. Follow the steps of Problems 4, 5, 6 above to decide what is the greatest possible difference between a true length and the measurement of a segment
 - a. $2\frac{1}{4}$ in. long, if measurement is to the nearest $\frac{1}{4}$ in.
 - b. $1\frac{3}{8}$ in. long, if measurement is to the nearest $\frac{1}{8}$ in.
8. Complete the following table from your answers to the exercises above.

When a segment is measured
to the nearest:

inch, $\frac{1}{2}$ in., $\frac{1}{4}$ in., $\frac{1}{8}$ in.

The greatest possible difference between the true
length and the measurement is:

? ? ? ?

The greatest possible difference between the true length of a segment and its measurement is called the greatest possible error in that measurement.

9. Does the table in Problem 8 show any relationship between the unit of measure and the greatest possible error? If so, what is it?

The greatest possible error in any measurement will always be half the smallest subdivision used in measuring. For example:

When a segment is measured to the nearest $\frac{1}{2}$ inch, the greatest possible error is $\frac{1}{2}$ of $\frac{1}{2}$ in. or $\frac{1}{4}$ in.

When a segment is measured to the nearest $\frac{1}{4}$ inch, the greatest possible error is $\frac{1}{2}$ of $\frac{1}{4}$ in. or $\frac{1}{8}$ in.

When a segment is measured to the nearest $\frac{1}{8}$ inch, the greatest possible error is $\frac{1}{2}$ of $\frac{1}{8}$ in. or $\frac{1}{16}$ in.

Figure 11-6b will help you see this.

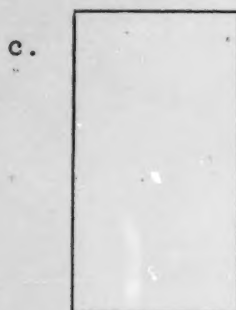
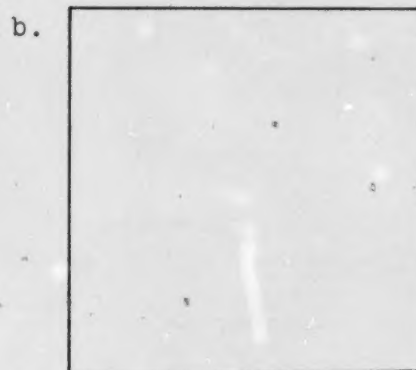
If we measure with a very small fraction of an inch, the possible error becomes very small. We cannot ever be sure of measuring any length exactly, however, because the act of measuring always means we have to decide whether the end of a segment is nearer one mark or another. For example, if we used a micrometer that measures to $\frac{1}{1000}$ of an inch, in reading the instrument we still have a possible error of $\frac{1}{2}$ of $\frac{1}{1000}$ in. or $\frac{1}{2000}$ inch.

A person who works with measurements must know what is meant by those measurements. He must know, for example, that when he writes the length of " $\overline{AB} \approx 6\frac{3}{4}$ inches" this means that $\overline{AB}$ was measured only to the nearest $\frac{1}{4}$ inch. The possible error therefore is $\frac{1}{2}$ of $\frac{1}{4}$ inch or $\frac{1}{8}$ inch, and the true length of $\overline{AB}$ is somewhere between $6\frac{5}{8}$ in. ($6\frac{3}{4} - \frac{1}{8}$) and $6\frac{7}{8}$ inches ($6\frac{3}{4} + \frac{1}{8}$).

Exercises 11-6b

1. Draw a segment 2 inches long and divide it so that it can be used to show a precision of $\frac{1}{8}$ inch.
2. Draw a segment 2 inches long and divide it so it can be used to show a greatest possible error of $\frac{1}{8}$ inch.

3. Measure the length and width of each of these figures to
 (1) the nearest $\frac{1}{2}$ inch; (2) the nearest $\frac{1}{8}$ inch; and
 (3) the nearest 16th inch. In each case give the
 greatest possible error.



4. Assume that the following measurements are written correctly
 (this means that fractions show the precision of measurement).
 Fill in the blanks as shown in part (a).

	Measurement	Precision of Measurement	Greatest Possible Error
a.	$3\frac{2}{4}"$	nearest $\frac{1}{4}"$	$\frac{1}{2}$ of $\frac{1}{4}" = \frac{1}{8}"$
b.	$1\frac{15}{16}"$		
c.	$4\frac{3}{8}"$		
d.	$2\frac{6}{8}"$		
e.	$3\frac{10}{16}"$		
f.	$7\frac{4}{8}"$		
g.	$2\frac{2}{32}"$	nearest $\frac{1}{32}"$	$\frac{1}{2}$ of $\frac{1}{32}" = \frac{1}{64}"$

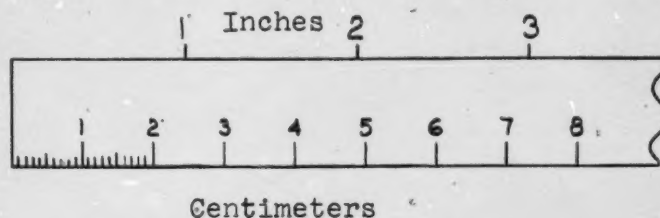
5. For each measurement given in Problem 4, state between what two lengths the real measurement lies. The answer for part (a) would be written: The real measurement lies between $(3\frac{2}{4} - \frac{1}{8})"$ or $3\frac{3}{8}"$ and $(3\frac{2}{4} + \frac{1}{8})"$ or $3\frac{5}{8}"$. The measurement for part (g) would be written: The real measurement lies between $(2\frac{3}{32} - \frac{1}{64})"$ or $2\frac{5}{64}"$ and $(2\frac{3}{32} + \frac{1}{64})"$ or $2\frac{7}{64}"$.
6. A rectangle has a length of $4\frac{0}{2}$ inches and a width of $3\frac{0}{2}$ inches.
- Draw a rectangle using the longest possible segments that have these measurements.
 - Inside the rectangle in (a) draw a rectangle that has the shortest possible segments with these measurements.
7. A square has sides of $3\frac{1}{2}$ inches with a greatest possible error of $\frac{1}{8}$ inch.
- Draw the largest square whose side has this measure
 - Inside the first square, draw the smallest square with this measure.
-

11-7. Metric Units of Length.

The metric system is the measuring system used in most of the countries of the world. It is not the official system of the United States and other countries where English is the principal language, but even in these countries the metric system is being used more and more. Let us see how measurement with metric units is done.

The meter is the principal metric unit of length. It is a little longer than a yard. It is a very convenient unit for measuring the lengths which we usually express in yards. In all track and field meets where athletes from several countries take part, distances are measured in meters.

When short lengths are measured, or when a more precise length is wanted, a smaller unit than the meter is needed. The most convenient unit is found by dividing a meter into 100 equal segments. Each of these units is called a centimeter (abbreviated cm.). Most rulers that you buy have centimeters marked along one edge. You should have one of these rulers to use.



If you examine such a ruler you will see that a centimeter is a little less than half an inch. You will also find that each centimeter is divided into ten parts, rather than 4, 8, or 16 as is done with inches. Parts of a centimeter can therefore be written with a decimal point as well as a common fraction.

Measure $\overline{RX}$ below to the nearest centimeter and to the nearest .1 cm. You should find that the length of $\overline{RX} \approx 9$ cm., or $\overline{RX} \approx 9.3$ cm.

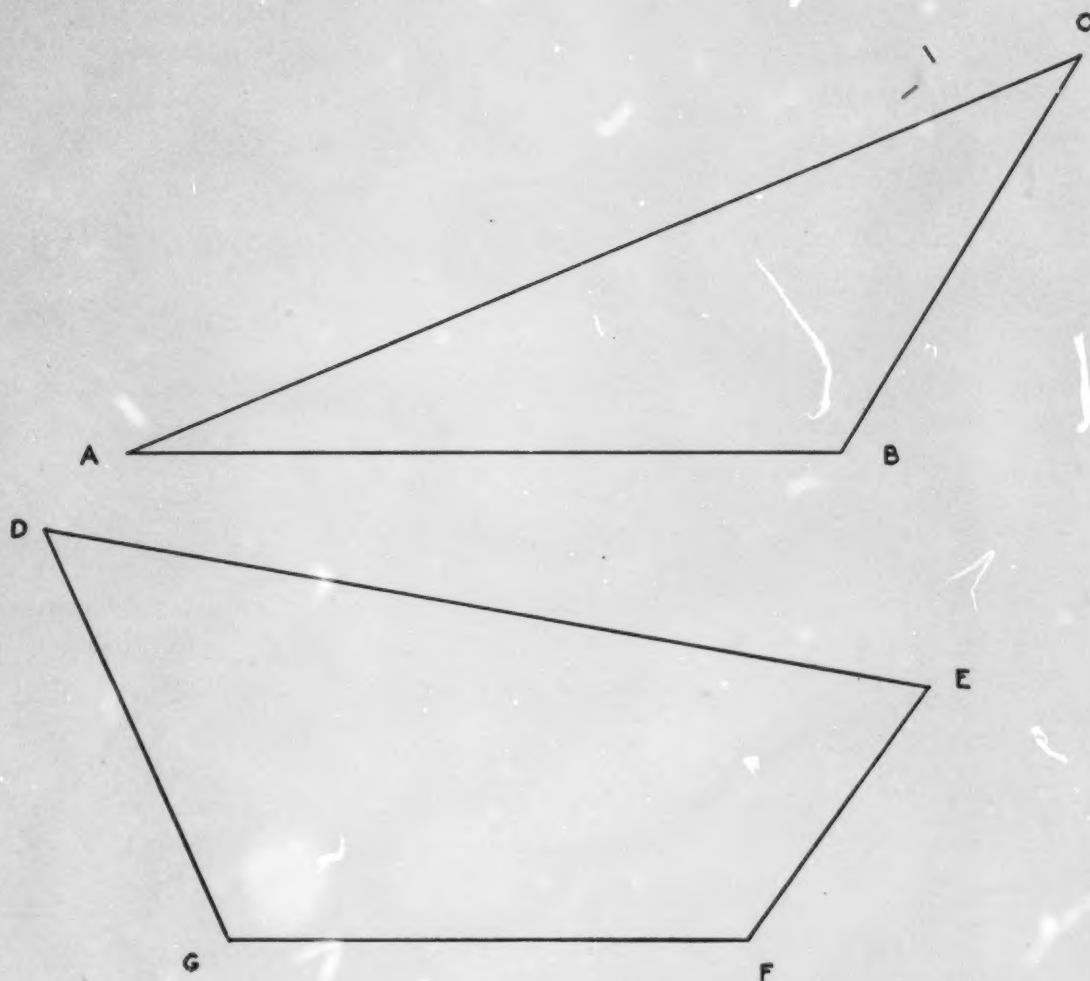
R ————— X

Exercises 11-7a

1. Draw segments of the following lengths. Make your drawings as precise as the marks on your ruler permit.

a. 4 cm.	c. 2 cm.
b. 6 cm.	d. 7 cm.
2. Measure the length and the width of the cover of your book. Measure each to the nearest centimeter and to the nearest .1 cm.

3. Measure the boundary line of each of the figures below. Measure each segment to the nearest centimeter and to the nearest .1 cm. When you record your measurements, remember the symbol " $\approx$ " should be used to show that these measurements are approximate.



4. Express the following as a number of centimeters. Use the method of Section 11-4d shown in the example below. The basic fact needed is: 1 meter = 100 cm.

$$\begin{aligned}\text{Example: } 2.3 \text{ meters} &= \underline{\quad ? \quad} \text{ cm.} \\ 2.3 \cdot (100 \text{ cm.}) &= \underline{\quad ? \quad} \text{ cm.} \\ &= \underline{230} \text{ cm.}\end{aligned}$$

- | | |
|---------------|----------------|
| a. 2 meters | e. .6 meters |
| b. 7 meters | f. .05 meters |
| c. 6.5 meters | g. .32 meters |
| d. 1.2 meters | h. 1.28 meters |
5. How many meters are in a segment of each of these lengths?
Use the method of Problem 4.
- | | |
|------------|------------|
| a. 300 cm. | e. 50 cm. |
| b. 700 cm. | f. 450 cm. |
| c. 256 cm. | g. 75 cm. |
| d. 185 cm. | h. 8 cm. |

In Problem 2, above, you measured segments with a precision of .1 cm. The name of this very small unit is a "millimeter." A millimeter is $\frac{1}{10}$ of a centimeter, which is $\frac{1}{100}$ of a meter. Therefore, a millimeter is $\frac{1}{10}$ of $\frac{1}{100}$, or $\frac{1}{1000}$ of a meter. "Millimeter" is abbreviated "mm" and "meter" is abbreviated "M". The facts you now know about metric units are in the table below.

1 cm. = $\frac{1}{100}$ M.	or	1 meter = 100 cm.
1 mm. = $\frac{1}{10}$ cm.	or	1 cm. = 10 mm.
1 mm. = $\frac{1}{1000}$ M.	or	1 meter = 1000 mm.

The fractions above can be written decimally if we prefer that form.

We often use millimeters in describing precise measures. You may be familiar with a camera that uses 16 mm. or 35 mm. film. Airplane rockets and other artillery weapons are measured in millimeters.

Class Exercises 11-7b

1. Draw segments of the lengths below. You will need to have a very sharp pencil and to mark very accurately in using such small units.

a. 20 mm.	e. 81 mm.
b. 50 mm.	f. 105 mm.
c. 70 mm.	g. 16 mm.
d. 35 mm.	h. 58 mm.
2. Exchange papers with a classmate and check the segments of Problem 1.

Exercises 11-7c

1. Measure in millimeters each segment of the figures in Exercises 11-7a.
2. Express these lengths in millimeters:

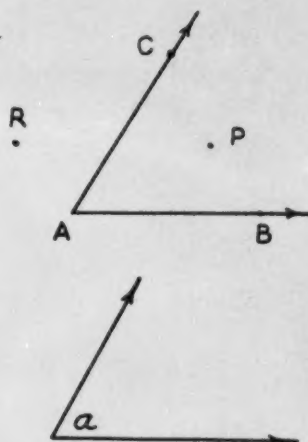
a. 3 cm.	e. 115 cm.
b. 12 cm.	f. 17.4 cm.
c. 2.8 cm.	g. 1 meter
d. 6.3 cm.	h. 3.5 meters
3. Express these in the units called for:

a. 40 mm. = _____ cm.	e. 156 mm. = _____ cm.
b. 100 mm. = _____ cm.	f. 2000 mm. = _____ cm.
c. 100 mm. = _____ M.	g. 2000 mm. = _____ M.
d. 32 mm. = _____ cm.	h. 204 mm. = _____ cm.

11-8. Measurement of Angles.

You have been studying ways of measuring line segments and surfaces. Now let us see how the interiors of angles are measured.

Recall that an angle is the set of points on two rays with the same end-point. In the drawing of the angle, the rays are $\overrightarrow{AB}$ and $\overrightarrow{AC}$. These rays are the sides of angle BAC and point A is its vertex. Notice that the angle is named "angle BAC" or "angle CAB," with the vertex named second. Why should A be the middle letter? We can call it "angle A" if this name applies to only one angle. We also name angles by writing a small letter or numeral in the interior of the angle, near the vertex.



Recall that an angle determines three sets of points in the plane:

- The set of points in the interior of the angle,
- The set of points in the exterior of the angle, and
- The set of points on the angle itself.

A point P is in the interior of $\angle BAC$ if:

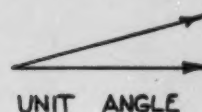
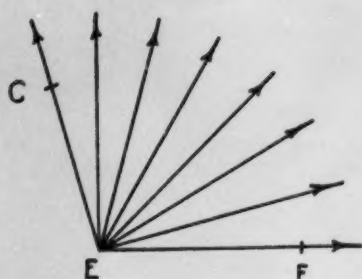
- (1) it is on the same side of line $\overleftrightarrow{AB}$ as point C,
- and (2) it is on the same side of line $\overleftrightarrow{AC}$ as point B.

In angle BAC above, the point P is in the interior of the angle. Any point which is not a point on the angle and is not in the interior of the angle must be a point in the exterior of the angle. Is point R a point in the exterior or the interior of angle BAC?

As you know, to measure anything you must use a unit of the same nature as the thing measured. To measure an angle, an

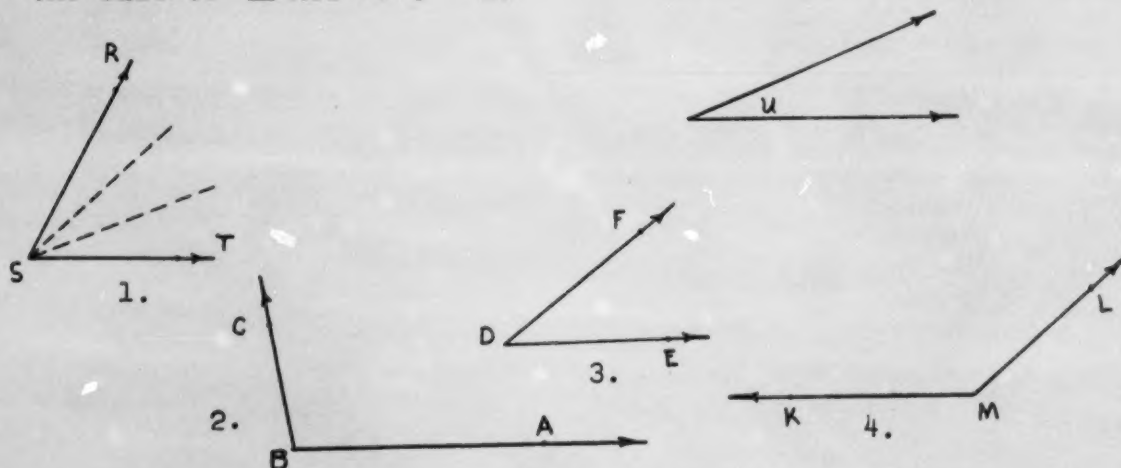
angle is chosen as the unit of measure. Then an angle can be measured by drawing rays which subdivide it into angles which are exactly like the unit angle. In the sketch, angle CEF is subdivided into angles which are like the unit angle shown. So the size of $\angle CEF$ is about 7 times the size of the unit angle. This may be written:

the size of $\angle CEF \approx 7 \cdot u$ (where u represents the measure of the unit angle.)



Exercises 11-8a

Let the angle marked "u" below be considered a unit angle. Copy $\angle u$ by tracing it on a piece of thin paper. Cut out the unit angle and use it to subdivide each of the other angles, as shown in angle TSR. Be sure that the vertex of u is always placed on the vertex of the angle measured. Compare each angle with the unit angle, stating your comparison like this: the size of $\angle RST \approx 3 \cdot u$.



5. a. Use the unit angle in Problem 1 to subdivide the exterior of angle RST in Problem 1. Use the same vertex you used to measure the interior of the angle. Then compare the exterior of angle RST with the unit angle. This gives the exterior measure of angle RST.
b. Subdivide the exterior of angle KML in Problem 4 and state its exterior measure.
 6. a. Which is larger: the measure of the interior of angle RST or its exterior measure?
b. What is the total of the interior measure and exterior measure of angle RST?
 7. a. Which is larger: the interior or exterior measure of angle KML?
b. What is the sum of the interior and exterior measures of angle KML?
-

Standard Unit for Angles

Just as there are standard units for measuring a line segment (inch, foot, yard) so are there standard units for measuring an angle. The one we shall use is determined by a set of three hundred sixty rays drawn from the same point. These rays divide the plane into 360 angles whose interiors are the same size. The standard unit is called a degree and is an angle whose interior is $\frac{1}{360}$ of a plane.

The rays are numbered in order from 0 to 359, forming a scale. There is a number for each ray, and also a ray for each number from 0 to 359. Not all 360 rays are shown in the sketch on the following page, but every tenth ray is drawn. Which ray might be thought of as corresponding to the number 360?

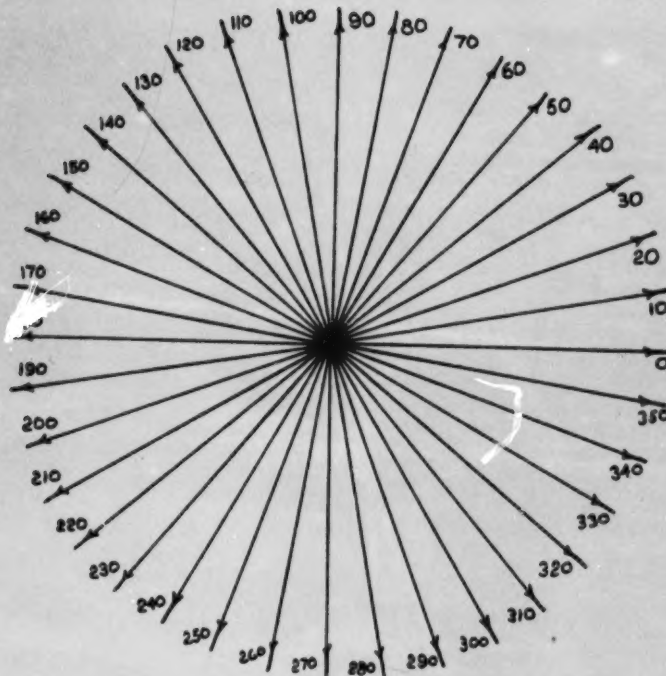


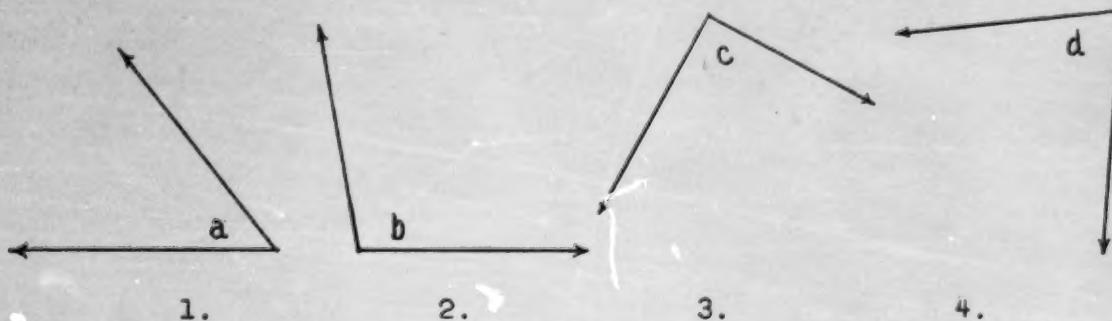
Figure 11-8a

You can use a scale like this to measure an angle. Place the angle on the scale with one side of the angle on the ray that corresponds to zero and the other side on a ray that corresponds to a number less than 180. Then the number which corresponds to that ray is the measure of the angle, in degrees.

The symbol for "degree" is " $^{\circ}$." Thirty-five degrees may be written " 35° ."

Exercises 11-8b

Copy the angles below by tracing them, and cut out your copies. Then measure each angle by placing it on the scale in Figure 11-8a.

The Protractor

The method you used for the exercises above is inconvenient, so usually an instrument called a protractor, which is half of the angle scale, is used. We use the protractor by placing it on the angle.

Look at this drawing of a protractor. Think of the rays from point *V*. Segments of these rays are shown on the curved part of the protractor. In the drawing, two of the rays are shown in dotted lines. These two rays correspond to 15° and 50° .

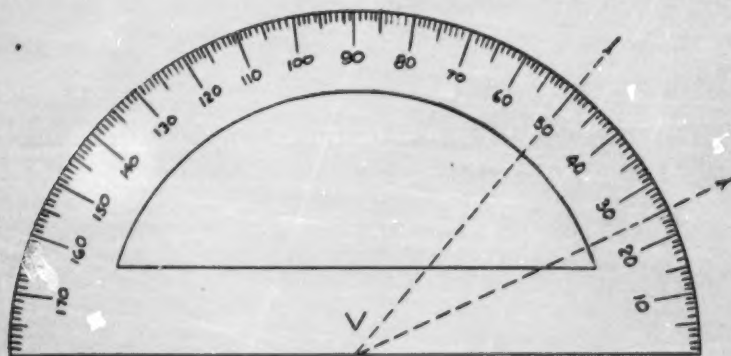


Figure 11-8b

To measure an angle with a protractor, do the following:

1. Place the protractor on the angle so that point V is on the vertex of the angle.
2. Be sure that the ray which corresponds to zero on the protractor lies on one side of the angle. (It does not make any difference which side you want to use.)
3. Now notice the protractor ray which is on the other side of the angle.
4. The number that corresponds to this ray is the measure, in degrees, of the interior of the angle.

You will find that your protractor may have two scales. One scale starts with zero at the right and runs to 180 at the left. The other scale starts with zero at the left and runs to 180 at the right. The zero may not be marked on your protractor, but you will be able to see where it is supposed to be. When you read the measure of an angle, you must be sure to read the same scale that shows zero for the first side of the angle.

Exercises 11-8c

1. In the drawing below is shown a protractor placed on a figure with several rays drawn from point A. Find the measure of each of the following angles:

- | | | | |
|-----------------|-----------------|-----------------|-----------------|
| a. $\angle BAK$ | d. $\angle BAH$ | g. $\angle GAM$ | j. $\angle CAG$ |
| b. $\angle BAC$ | e. $\angle BAE$ | h. $\angle MAC$ | k. $\angle KAF$ |
| c. $\angle BAD$ | f. $\angle MAF$ | i. $\angle DAE$ | l. $\angle HAF$ |

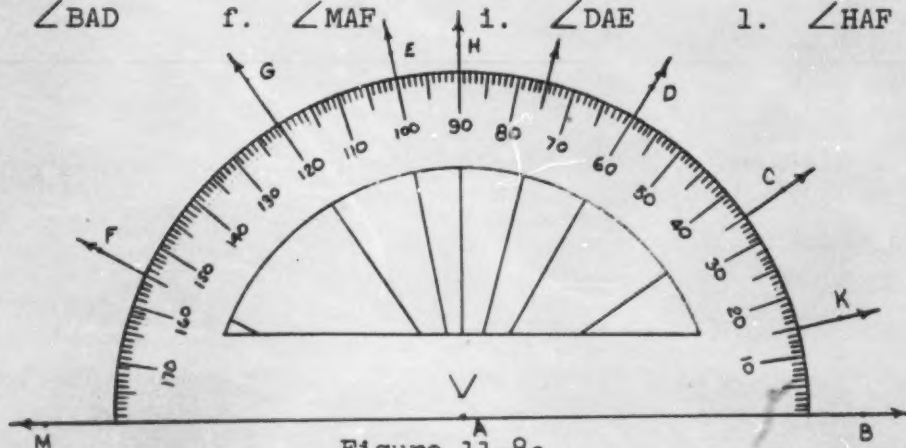


Figure 11-8c

2. Use a protractor to measure the angles below. If the parts of the rays shown are not long enough to show the intersection of the side of the angle with the rim of the protractor, lay the edge of a piece of paper along the side of the angle.



3. Can you use your protractor to find the exterior measure of an angle? If so, find the exterior measure of the angles in Problem 2.
4. Draw a ray $\overrightarrow{AB}$ with end-point A. Place your protractor with point V on A and the protractor ray which corresponds to 0° on $\overrightarrow{AB}$. Then mark the point at 35° on the protractor scale, and name it point C. Remove your protractor and draw ray $\overrightarrow{AC}$. You should now have an angle BAC. What is its measure?
5. Use the method described in Problem 4 to draw angles of these sizes.
- | | |
|---------------|----------------|
| a. 20° | d. 90° |
| b. 45° | e. 130° |
| c. 61° | f. 179° |
6. Can you think of a way to find the exterior measures of the angles in Problem 5 without using your protractor? If so, find them.

7. In the drawing below, ray $\overrightarrow{AB}$ and ray $\overrightarrow{AC}$ are opposite rays; that is, they are on the same line, they have the same end-point, and their intersection set is the end-point A.



If the protractor is placed so that V is on A and the zero ray of one scale is on ray $\overrightarrow{AB}$, what number corresponds to the protractor ray on ray $\overrightarrow{AC}$?

Sets of Angles.

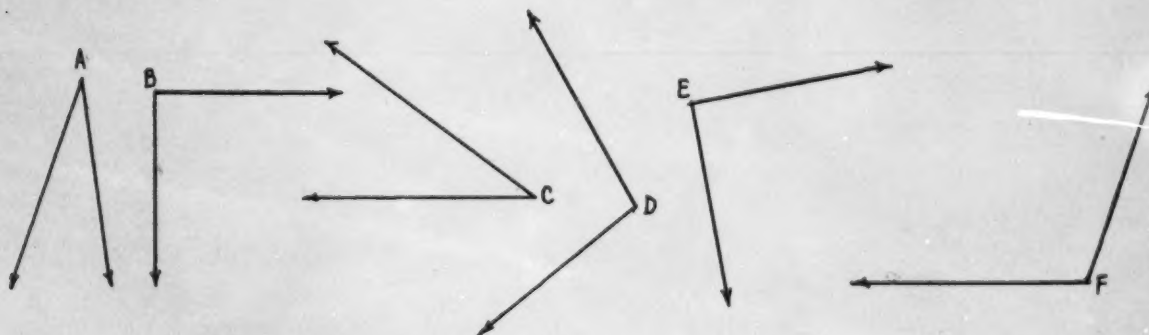
Angles are separated into sets according to their measures.

Draw an angle whose size is 90 degrees. An angle of 90 degrees is a right angle.

Draw an angle whose size is less than 90 degrees. Such an angle is called an acute angle. Draw an angle whose size is more than 90 degrees and less than 180 degrees. It is called an obtuse angle.

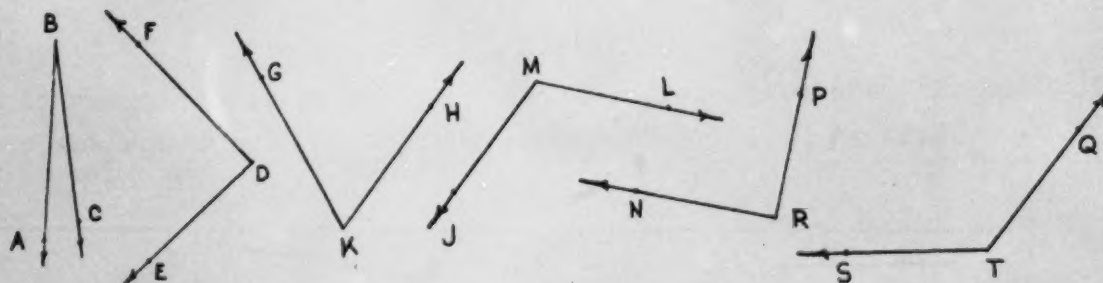
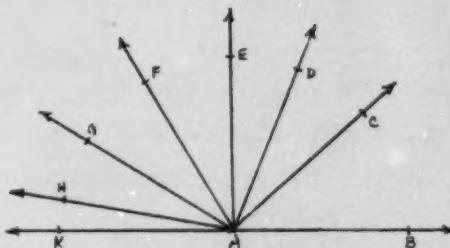
Exercises 11-8d

1. Without measuring, tell which of the angles below appear to be:
 a. right angles; b. acute angles; c. obtuse angles.



2. Were you uncertain about any of the angles in Problem 1? If so, check your answer by using a protractor.

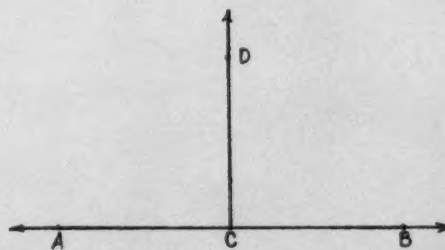
3. a. The measure of an acute angle is greater than ? and less than ?.
 b. The measure of an obtuse angle is greater than ? and less than ?.
4. a. In the figure, name all the obtuse angles which have ray $\overrightarrow{AB}$ as one side; all such acute angles; all such right angles.
 b. Name all the acute angles that have ray $\overrightarrow{AE}$ as one side; all such obtuse angles; all such right angles.
 c. Name all the right angles that have ray $\overrightarrow{AK}$ as one side; all such obtuse angles; all such acute angles.
5. a. Without measuring, tell whether each angle below is an acute, right, or obtuse angle.
 b. Without measuring, estimate the number of degrees in the interior of each angle. (A good way to do this is to think how it compares with a right angle.)



- c. Measure each angle in Problem 5b. How well did you estimate their sizes?
6. Find six examples of our idea of an angle--to acute, two right, and two obtuse angles.

Perpendicular Lines.

In the figure, ray $\overrightarrow{CD}$ is drawn from point C on line $\overleftrightarrow{AB}$, so that angle BCD and angle DCA have the same measure. We say that ray $\overrightarrow{CD}$ is perpendicular to line $\overleftrightarrow{AB}$.

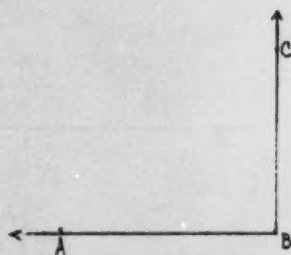


If a protractor is placed with point V on point C and the zero ray of the protractor on $\overrightarrow{CB}$, the protractor ray which corresponds to ? will fall on $\overrightarrow{CA}$. Therefore the measure of angle BCD, and also of angle DCA is ? degrees, and they are both ? angles. We may also say that a ray is perpendicular to a line if the ray and the line intersect so that at least one right angle is formed.

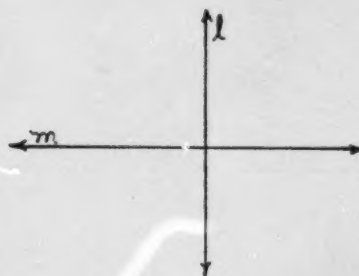
When two rays form a right angle we say the rays are perpendicular. Which rays in the figures for Problem 5 above are perpendicular rays?

Two lines are perpendicular if rays determined by the point of intersection of the lines form a right angle. And line segments are perpendicular if the lines on them are perpendicular.

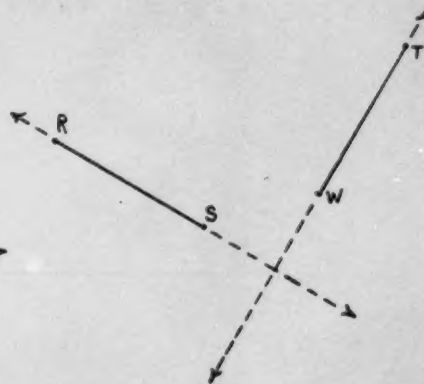
The symbol for "perpendicular" is " $\perp$."



$\overrightarrow{BA} \perp \overrightarrow{BC}$.

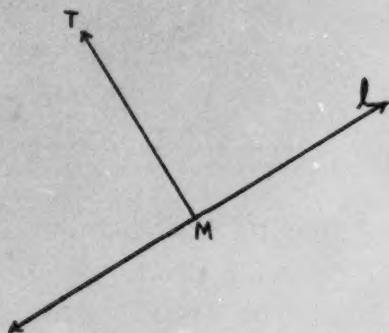


Line $l \perp$ line m .

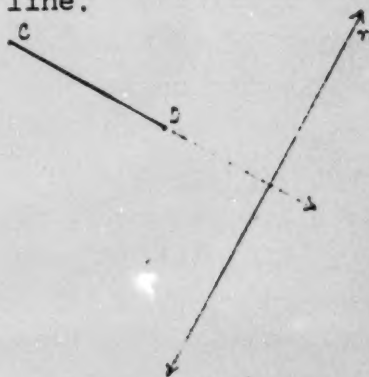


Segment $\overline{RS} \perp$
segment $\overline{TW}$.

We may also say a line is perpendicular to a ray, or a line segment is perpendicular to a ray or a line.



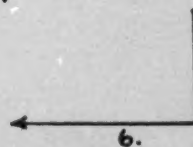
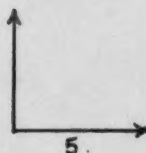
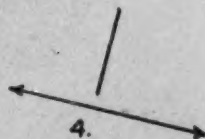
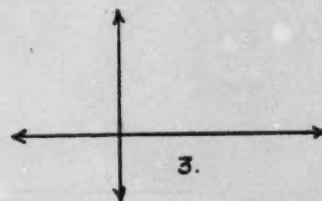
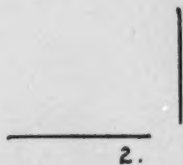
Line $l \perp$ ray $\overrightarrow{MT}$.



Segment $\overline{CD} \perp$ line n .

Exercises 11-8e

1. Which of the drawings below represent
 - a. Perpendicular lines?
 - b. Perpendicular rays?
 - c. Perpendicular line segments?
 - d. A line perpendicular to a ray?
 - e. A ray perpendicular to a line segment?
 - f. A line segment perpendicular to a line?



2. Find five examples of pairs of perpendicular rays, lines, or line segments.
3. Find five examples of pairs of rays, lines, or line segments which are not perpendicular. If they intersect, tell whether the rays on them form an acute angle or an obtuse angle.

11-9. Summary.

1. The size of collections of separate objects may be found by counting, but the size of continuous quantities is found by measuring.
2. In practice measurement is approximate, not exact. Measurement of the continuous geometric quantities of length and angle may be thought of as a process of "covering" with units of a given size.
3. When geometric continuous quantities are measured, the unit used must be of the same kind as the quantity measured. For example, a unit segment is used to measure segments, a unit angle to measure angles.
4. Units of measure can be of any convenient size, but in practice it is important to have standard units which are agreed upon by large groups of people.
5. The symbol $\approx$ means "is approximately equal to."
6. "Precision" of a measurement depends on the size of the unit, or subdivision, used in making the measurement. A length measured in inches is more precise than one measured in feet. The precision of a measurement should be shown in the way it is written.
7. Linear measure is measure of length.
8. The greatest possible error in measurement of a segment is the possible difference between the stated measurement and the true length of the segment.
9. The same measurement may have many different names, for example, 108 in., 9 ft., 3 yd., $\frac{3}{1760}$ mi. One name may be changed for another by making use of the ratios between the units of measure.
10. A ruler may be used as a number scale to measure segments, and a protractor may be used as a number scale to measure angles.

11. Angles are measured in degrees. An angle of 90° is a right angle. An angle of less than 90° is an acute angle; one of more than 90° but less than 180° is an obtuse angle.
12. Lines, segments, or rays that form a right angle at their intersection are perpendicular.
13. Many quantities other than geometric are measured, frequently by reading linear or circular scales. Thus measurement of them is similar to the kinds of measurement discussed in this chapter.

11-10. Chapter Review.

Exercises 11-10

1. Which of the following can be counted? Which must be measured?
 - a. the length of this page
 - b. the length of a school day
 - c. your age
 - d. the number of periods in a school day
 - e. the school enrollment
 - f. the amount of gasoline used in a car
2. Measure the following segments to the nearest $\frac{1}{4}$ inch and nearest $\frac{1}{8}$ inch. Write the measurements so they show
 - a. that the answer is only approximate.
 - b. the precision used in measuring.

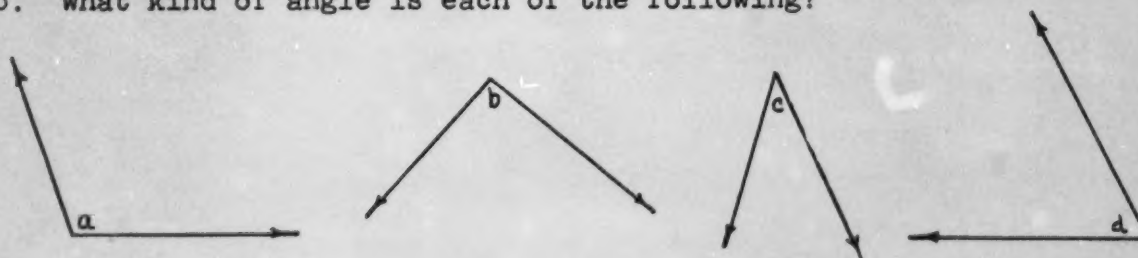
A _____ B

C _____ D
3. Measure the segments above to the nearest
 - a. centimeter,
 - b. millimeter.

4. Which of the following are continuous quantities?
- Population of a town
 - Distance to town
 - Size of a family
 - Interior of an angle.
5. Arrange the terms "acute angle", "obtuse angle", "right angle" in the following sentence. There will be one phrase left for you to supply.

_____ > _____ > _____ > _____

6. What kind of angle is each of the following?



7. Measure each of the angles above. Extend the sides if necessary.
8. Name six units of linear measure.
9. Name four standard units that are not linear units.
10. Complete: If the length of a segment is correctly stated to be $8\frac{3}{4}$ inches, it might be as short as _____, or as long as _____.
11. Which of these would not be an appropriate unit with which to measure a distance?
- a lollipop stick
 - a child's building block
 - a piece of string
 - a cup
 - the end section of your thumb
12. Change these measurements to the units asked for.
- | | |
|-----------------------|------------------------|
| a. 3.5 M. = _____ cm. | e. 24 cm. = _____ M. |
| b. 13 ft. = _____ in. | f. 144 in. = _____ yd. |
| c. 8 in. = _____ ft. | g. 1760 yd = _____ mi. |
| d. 24 cm. = _____ mm. | h. 750 mm. = _____ M. |

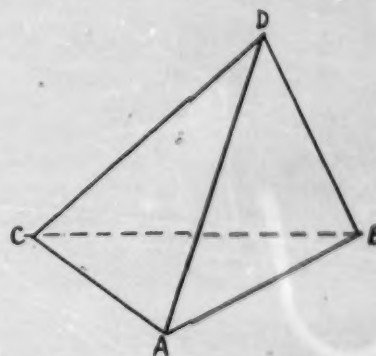
11-11. Cumulative Review.Exercises 11-11

How Are You Doing?

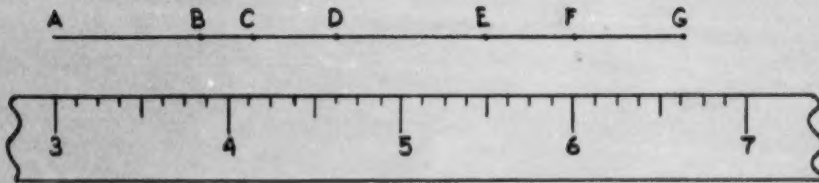
- Use the distributive property to rewrite each of the following.
 - $9 \cdot (3 + 2)$
 - $(7 + 11) \cdot 6$
- Are the following sets closed under addition?
 - $\{2, 4, 6, 8, 10, \dots\}$
 - $\{7, 14, 21, 28, 35\}$
- Copy and replace the ? by < or > whichever is correct.

a. $6 ? 8$	d. $\frac{13}{15} ? \frac{5}{6}$
b. $\frac{1}{2} ? \frac{9}{17}$	e. $\frac{3}{4} ? \frac{25}{40}$
c. $\frac{9}{14} ? \frac{20}{31}$	f. $\frac{15}{18} ? \frac{28}{32}$
- In which base is 203152_7 written?
 - Base five
 - Base ten
 - Could be any base less than 5
 - Could be any base greater than 5.
- Which numeral represents the largest number?

a. 28_{ten}	c. 10000_{two}
b. 23_{twelve}	d. 37_{eight}
- In the diagram on the right
 - How many planes contain point D? Name them.
 - How many planes contain $\overleftrightarrow{AC}$? Name them.
 - What is $\overleftrightarrow{AC} \cap \overleftrightarrow{AB}$?
 - Are $\overleftrightarrow{CA}$ and $\overleftrightarrow{AD}$ in the same plane? State the property that says this.



7.



How long is each segment? Measure to the nearest 16th.

a. $\overline{AB}$

d. $\overline{AE}$

b. $\overline{AC}$

e. $\overline{AF}$

c. $\overline{BD}$

f. $\overline{EG}$

8. Perform the indicated operations and simplify.

a. $5\frac{3}{5} + 4\frac{4}{5}$

d. $3\frac{7}{8} - 1\frac{3}{8}$

b. $1\frac{1}{4} + 2\frac{2}{3}$

e. $5\frac{1}{6} - 2\frac{5}{6}$

c. $2\frac{1}{4} + \frac{3}{8} + 1\frac{1}{6}$

f. $1\frac{1}{3} \times \frac{3}{4}$

9. Write a complete factorization of 66.

10. Which of the statements below is not true?

a. If a number is divisible by 3 and 5 then it is also divisible by 15.

b. A multiple of 6 must be an even number.

c. The number 47 is a composite number.

11. If a TV antenna tower has a shadow 275 feet long when a 6 foot man's shadow is 15 feet long, find the height of the tower.

12. Joan reduced her weight during the summer from 150 pounds to 120 pounds. What was the per cent of decrease?

Chapter 12
AREA AND VOLUME

12-1. Rectangle.

We will consider here the most familiar of the simple closed curves, the rectangle. Let us agree that a rectangle is a four sided figure (in a plane) which has a right angle at each of its four corners. Is the cover of your book in the shape of a rectangle? Find five examples of rectangles in your classroom. Under this definition is a square a rectangle?

In Chapter 11 you solved problems in measuring simple closed curves. You learned to measure the lengths of the segments forming the curves, and you also learned to compare the closed regions determined by the curves. What instrument did you use to measure the lengths of the sides? Why couldn't you use a ruler to measure the closed region?

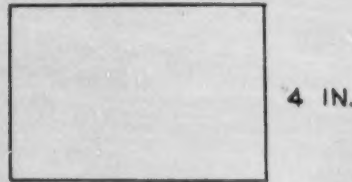
The total length of a closed curve is called its perimeter. For a rectangle, this is the distance an ant would walk if it started at one corner of the rectangle and kept walking along the sides till it returned to its starting point.

Class Exercises 12-1a

1. Measure the four sides of
 - a. the front cover of this book.
 - b. a window pane in your classroom.
 - c. a section of chalkboard.
2. Find the perimeters of each of the three rectangles in Problem 1.
3. In any rectangle there are two pairs of opposite sides, that is, sides that do not meet at a corner. For each rectangle in Problem 1 write down the lengths of the pairs of opposite sides. From looking at these lengths complete the following statement.

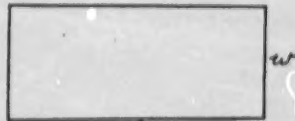
The lengths of two opposite sides of a rectangle seem to be

4. Suppose two sides of a rectangle are 6 inches and 4 inches as shown:



Use the result of Problem 3 to tell the lengths of the remaining sides. Find the perimeter of this rectangle.

5. If l and w stand for the numbers of units in the lengths of two sides of a rectangle, what are the numbers of units



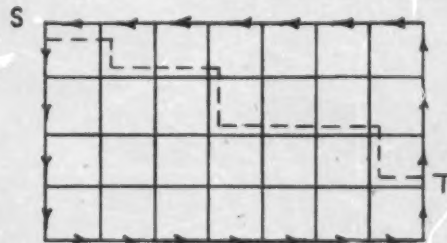
in the lengths of the other sides? Write a number sentence which tells how to find the number of units in the perimeter if you know l and w

The lengths of two intersecting sides of a rectangle are called the length and width of the rectangle. The total distance around a rectangle is called the perimeter. Quite often the total distance around other simple closed curves is merely called the length of the curve. For a rectangle, the word length indicated the length of the longer side. We will use it here in that way.

Exercises 12-1b

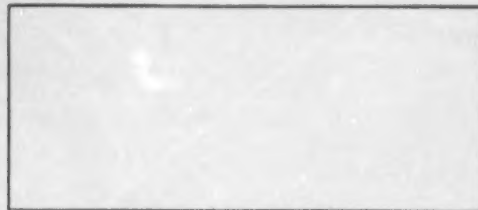
1. A school playground is in the shape of a rectangle 400 ft. long and 200 ft. wide. Make a drawing to represent the playground. What is the total length of the fence around it?
2. If fencing costs \$5 a yard, how much did the fence of Problem 1 cost?

3. A carpenter is nailing picture molding around a room which has a length of 15 ft. and a width of 10 ft. How much molding will he need? Express the answer in feet, and in yards.
4. The carpenter of Problem 3 is also putting in a baseboard around the room. He notices that the room has four doorways, and that each doorway is 3 ft. wide. Since he does not put baseboard across the doorways, how many feet of baseboard will he need? (Does it matter where the doorways are located?)
5. A boy has 24 ft. of wire fence to make a rectangular pen for his pet rabbit. He plans to use all the fence in making the pen.
 - a. Could he make a pen 12 ft. long and 12 ft. wide? Why or why not?
 - b. Could he make a pen 8 ft. long and 3 ft. wide?
 - c. How about 8 ft. long and 4 ft. wide?
 - d. Give five examples of lengths and widths he could use for his pen. (Use only whole numbers for lengths and widths.)
6. A girl is decorating for a party. She has 5 tables, each 28 inches wide and 42 inches long, and wants to put a strip of crepe paper around the edge of each table.
 - a. How many inches of crepe paper are necessary?
 - b. How many yards of crepe paper are needed?
7. A Fourth of July parade is to follow the route shown by the arrows, starting at S. The squares represent city blocks.



In this city each block is $\frac{1}{8}$ mile on a side. What is the total length of the parade route?

8. If the decorations along the parade route in the parade of Problem 7 cost about \$250 a mile, what was the approximate total cost of the decorations?
9. In the parade of Problem 7 one man got tired, and from point T, he sneaked back to S by the route shown by the dotted lines.
 - a. How far did he travel?
 - b. How much distance did he save?
10. Measure, in centimeters, the length and width of the rectangle below.
 - a. The length is _____ cm.
 - b. The width is _____ cm.
 - c. The perimeter is _____ cm.



11. Repeat Problem 10 but measure the length and width in millimeters.
 - a. The length is _____ mm.
 - b. The width is _____ mm.
 - c. The perimeter is _____ mm.
12. A farmer found that it took 240 feet of fence to go around his rectangular farmyard. He noticed that one of the sides was 40 feet long. Find the length of each of the other sides.

12-2. Areas of Rectangles.

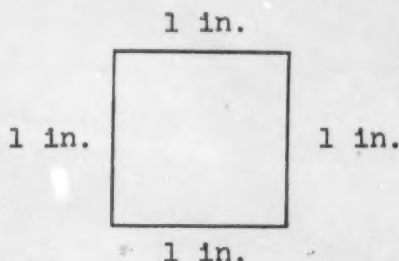
In Chapter 7 we learned that the interior of a simple closed curve together with its boundary is called a closed region. Let

us agree that when we say

- a. "area of a rectangle" we mean the area of the closed region determined by a rectangle.
- b. "area of a square" we mean the area of the closed region determined by a square.
- c. "area of a circle" we mean the area of the closed region determined by a circle.

With this idea in mind, let us turn our attention to the area of a rectangle. (Remember we are talking about the area of the closed region determined by a rectangle. A rectangle itself is just a set of points. Without our agreement it would be nonsense to speak of the area of the set of points called a rectangle.) To measure this area, we could choose the area of some simple closed curve for a unit of area. Then we can see how many of these units it takes to cover the closed region determined by the figure. This is an awkward and slow process.

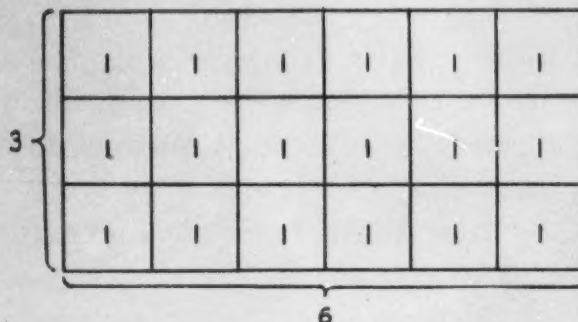
We would like to choose a definite unit for area. The usual choice is a square closed region. We have a choice as to how large a square to use. Let us take for the unit of area a square whose side is one unit of length. Since we have used many units of length, we shall have many units of area. If length is measured in inches, we have as the corresponding unit of area, the area of the square shown below.



The area of this square is called a square inch. Describe and name three other units of area related to different units of length.

If a rectangle is 6 units long and 3 units wide, then it

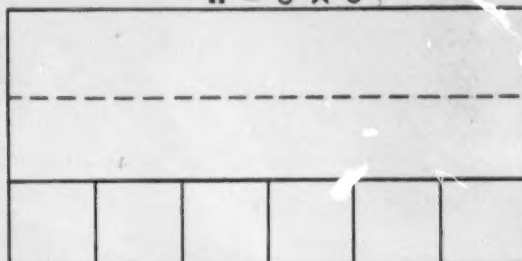
is clear at once that the area of this rectangle can be covered by square units of area as shown.



The measure of the area is, then, by definition, the number of squares. How many squares are there? Is there an easier way to get the number than by counting them? If so, how?

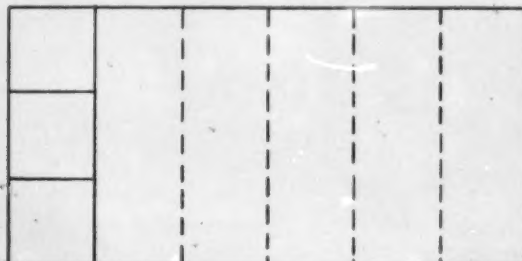
If you counted by noticing that each row has six squares and that there are three rows, you obtained the number of square units A in the area by writing:

$$A = 3 \times 6$$



If you counted by saying that there are three squares in each column and six columns you obtained:

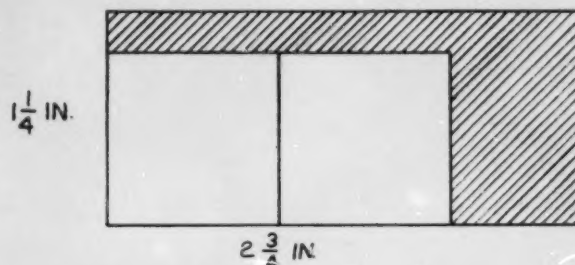
$$A = 6 \times 3$$



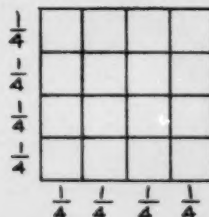
Are these answers the same? If so, what property of rational numbers does this illustrate? What advantage do you see in having the unit of area chosen so that the sides of the square are each

one unit of length? What if the unit of area were chosen as the area of a square whose side was 2 units? $1\frac{1}{2}$ units?

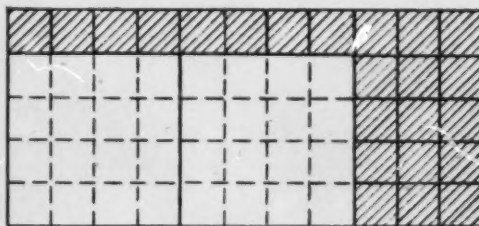
In practice it is very likely that the lengths of the sides will be rational numbers which are not counting numbers. Suppose for instance that the length is $2\frac{3}{4}$ inches and the width is $1\frac{1}{4}$ inches. We can easily put in two of the square inch units but are still left with a border as shown by the shaded area. In order to fill this border let us take a unit square inch and



cut it up into smaller squares as shown.



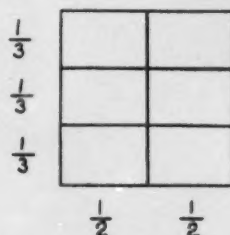
This particular division is chosen because the measurements given for the rectangle are in quarter inches. How many of the small squares are there in the square inch? Each small square is $\frac{1}{16}$ of a square inch. These small squares can be used to fill out the uncovered border.



As a matter of fact, if we cut up the original two square inch units also, as shown by the dotted lines, the entire rectangle to be measured is covered by these small squares.

Class Exercises 12-2a

1. In the above figure showing the rectangle $2\frac{3}{4}$ inches by $1\frac{1}{4}$ inches:
 The number of small squares in each row is ?; the number of rows is ?; the total number of small squares is ?; the area of each small square is ? square inches.
 The area of the rectangle is ? square inches.
2. a. In the problem was it necessary to draw the figure to find how many squares were in each row?
 b. To find the number of rows?
 c. If not, how do you determine these numbers?
3. The given length and width of a rectangle are $5\frac{1}{2}$ inches and $4\frac{1}{2}$ inches.
 a. A small square of what size would be convenient in covering this rectangle?
 b. How many squares are in each row? How many rows are there?
 c. What is the number of square inches in the area?
 d. Draw a figure showing the small squares.
 e. Was it necessary to draw the figure to find the area?
- *4. Using the same method as above, find the area of a rectangle whose length and width are $5\frac{1}{2}$ inches and $4\frac{2}{3}$ inches. (In this case you may find it easier to cut the unit square inch into rectangles instead of squares by making two divisions horizontally and three vertically as shown.)



Why might this division occur to you? What part of the square inch is the area of each small rectangle?

5. From your experience with these problems, state in words a method by which you can find the number of square units in the area of a rectangle if you know the number of units in the length and width.
 6. If l and w stand for the number of linear units in the length and width of a rectangle, and A is the number of square units in its area, write a number sentence telling how to find A if you know l and w . Notice that this is just the translation into mathematical language of the statement in Problem 5 above.
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In the exercises below we are going to deal with standard units and their relations. It may be helpful if we list some of the relations which you studied in Chapter 11.

$$12 \text{ inches} = 1 \text{ foot}$$

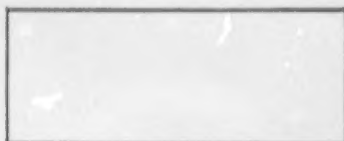
$$3 \text{ feet} = 1 \text{ yard}$$

$$36 \text{ inches} = 1 \text{ yard}$$

Exercises 12-2b

1.
 - a. How many square inches are in a square foot?
 - b. How many square feet in a square yard?
 - c. Draw figures to illustrate your answer.
2. Draw a square 3 inches on a side. Draw also a rectangle whose area is 3 square inches.
 - a. Which is larger?
 - b. What is the area of the square?
3. Draw two different rectangles, each of which has an area of 1 square inch. Make one rectangle 2 inches long and the other 4 inches long. (This shows how an area of 1 square inch may occur in many different shapes.)
4. A living room rug is 9 ft. by 12 ft. Find its area. Express the result in square feet and also in square yards.

5. A baseball diamond is a square, 90 ft. on a side. Find the area of this diamond both in square feet and in square yards.
6. The diamond used in softball is a square 60 ft. on a side. Is this area more than or less than half the area of a baseball diamond? (See exercise 5 above.)
7. A square is 10 mm. on a side.
- Find the area of the square in square millimeters.
 - Express your answer in square centimeters.
8. Measure in centimeters the length and width of the rectangle below.
- The length is _____ cm.
 - The width is _____ cm.
 - The area is _____ square cm.



9. Repeat exercise 8, but measure the length and width in millimeters.
- The length is _____ mm.
 - The width is _____ mm.
 - The area is _____ square mm.
10. Two rectangles are placed together as shown to form a larger rectangle. The number of linear units in the sides is indicated. Find the areas of both smaller rectangles and of the larger one in square units.
- Is the larger area the sum of the other two?
 - What property of rational numbers does this suggest?



11. A rectangle is 3 units long and 2 units wide.
 - a. If another rectangle is twice as long but has the same width, how do the areas of the two rectangles compare?
 - b. Draw a figure illustrating your answer.
 - c. Do the same if the new rectangle has the same length as the original one but twice the width.
12. Does the reasoning of the last problem depend upon the particular measures 3 and 2? If not, write a statement telling the effect on the area of any rectangle if you double the length and width.
13. If a rectangle has length and width of 3 units and 2 units, what is the effect on the area of doubling both length and width? Draw a figure to illustrate your conclusion. Write a statement telling the effect on the area of doubling both length and width of any rectangle.
14. In the rectangles of Problem 13, what is the ratio of the larger perimeter to the smaller perimeter?

12-3. Precision and Error.

In our work so far we have assumed that the lengths and widths of the rectangles are exact. Actually, we have seen that this is almost never the case because no measurement can be made exactly. If we measure a rectangle and find measurements of $3\frac{3}{4}$ inches and $2\frac{1}{4}$ inches, we should use the "approximately equal" symbol and write $l \approx 3\frac{3}{4}$, $w \approx 2\frac{1}{4}$ and therefore:

$$A = lw$$

$$A \approx (3\frac{3}{4})(2\frac{1}{4})$$

$$A \approx (\frac{15}{4})(\frac{9}{4})$$

$$A \approx \frac{135}{16}$$

$$A \approx 8\frac{7}{16}$$

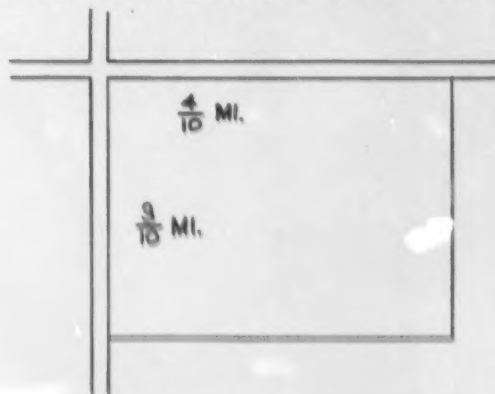
Since A is the number of square inches, we find that the area is approximately $8\frac{7}{10}$ square inches.

A statement concerning a measured quantity should indicate that it is only approximate, and not an exact measure.

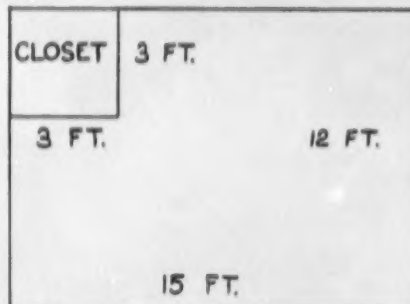
Exercises 12-3

Use the $\approx$ sign in connection with numbers representing measured quantities.

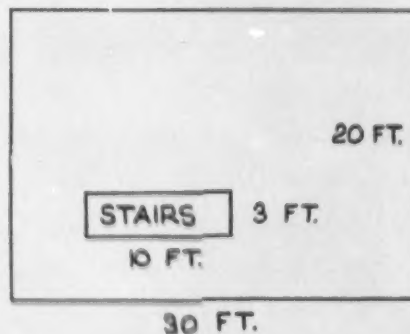
1. Measure the length and width of the top of your desk to the nearest half inch.
 - a. Find the number of square inches in the area.
 - b. What is the perimeter?
2. A section of chalkboard is about 5 feet long and $3\frac{1}{2}$ feet wide.
 - a. Find the area. Express the answer in square feet and square yards.
 - b. Find the perimeter and express it in feet, and yards.
3. If a garage floor is measured to be 18 feet by 22 feet
 - a. Find the area of the garage floor.
 - b. Find the cost of laying a concrete floor at 60 cents a square foot.
4. A rectangular field is located at the intersection of two perpendicular roads. The length and width are measured as approximately $\frac{3}{10}$ mile and $\frac{4}{10}$ mile. Find the area of the field. Express the result in square miles.



5. A rectangular lawn is found to be 84 feet by 50 feet. The lawn is to be seeded with grass. The directions on the box of grass seed say that one pound of seed is enough for 300 square feet. How many pounds of grass seed are necessary?
6. A bathroom floor is tiled with small tiles which are one inch squares. The floor contains 3240 of these tiles.
- What is the area of the floor in square inches?
 - What is the area of the floor in square feet?
7. A boy's room is in the shape of a rectangle. The length and width are measured as 15 feet and 12 feet. There is a closet 3 feet long and 3 feet wide built in one corner, as shown in the floor plan. What is the floor area of the room (not counting the closet)?



8. An attic is in the shape of a rectangle with measurements of 30 feet by 20 feet. There is an opening in the floor as shown where the stairway comes up.
- Find the floor area of the attic.
 - Does it matter in figuring area, where the opening for the stairs is placed?



12-4. Rectangular Prism.

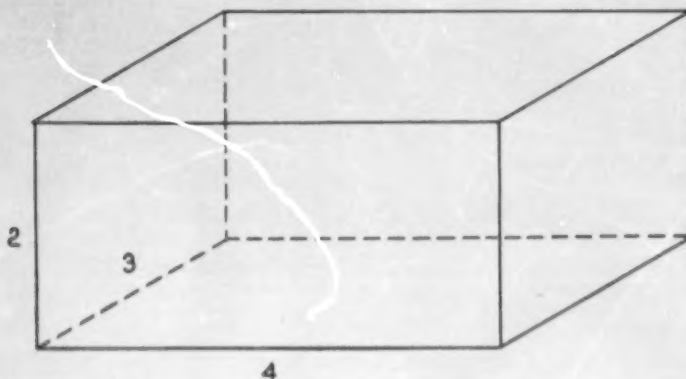
We call a figure like a chalk box a rectangular prism. The rectangular prism is a familiar figure and you can find many examples. Your classroom is probably one example. Name as many examples as you can.

When you walk across the classroom floor you are moving in the interior of the rectangular prism of your classroom, if your room has this shape. Let us examine such a prism. Note that the prism has a certain number of flat sides. These are called its faces. How many faces does a rectangular prism have? What kind of figure is each of the faces? Notice that each of the faces lies in a plane. For each face of the prism there is just one other face that does not meet it. Such a pair of faces are called opposite faces. Opposite faces actually lie in parallel planes. How many pairs of opposite faces are there? Identify the pairs of opposite faces in your classroom. What can you say about the shape of two opposite faces? How do you know?

You learned in Chapter 4 that two planes which intersect must intersect in a line. Then two faces which are not opposite must intersect in points that lie on a line. These segments are called the edges of the prism. How many edges are there on a rectangular prism? Identify them in your classroom. Some of these edges have the same length. What sets of edges are equal? Why? How many different lengths could there be among the edges? There are certain points on the prism where three faces intersect. These points are called vertices of the prism. (One such point is called a vertex.) How many vertices are there? Point out the vertices of your classroom.

As you have probably noticed, the parallel edges of a rectangular prism have the same length. For example, the edges from the bottom to the top have the same length. The edges from one end to the opposite end have the same length. There can be at most three different lengths.

In the figure below, the number of units in the lengths of three edges have been marked. Tell the number of units in the length of each of the other nine edges.



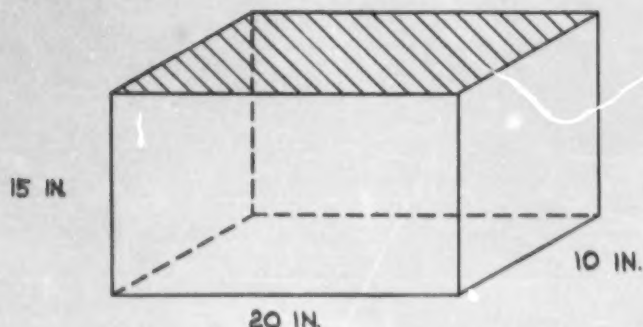
Area of a Prism.

The lengths of the edges in the three possible directions are often called the length, width, and height of the prism. What do you notice about the pairs of opposite faces? Since all the faces are rectangles, it is easy to find all their areas. The sum of the areas of all the faces is called the surface area of the rectangular prism.

Exercises 12-4

1. Find the surface area of the rectangular prism represented above.
2. A housewife has a cake tin in the shape of a rectangular prism (without a top). The tin is 10 inches long, 8 inches wide, and 2 inches deep. When she bakes a cake she lines the pan with wax paper. How many square inches of wax paper are needed to cover the inside of the tin?
3. A classroom wall is 30 feet by 10 feet. On this wall is a chalkboard 20 feet long and $3\frac{1}{2}$ feet wide. The wall, except for the chalkboard, is to be painted. What is the area to be painted? Express the answer in square feet, then in square yards.

4. An incubator for hatching eggs is in the shape of a rectangular prism 20 inches long, 10 inches wide, and 15 inches high. The top is made of glass (indicated by the shading).



The sides and bottom are made of wood. What is the area of the piece of glass used? What is the area of the outside of the wooden box? Express your answers both in square inches and in square feet.

5. A room is 15 feet long, 12 feet wide, and 9 feet high.
- How many squares of tile, each 12 inches by 12 inches, will be needed to tile the floor?
 - How many tiles will be needed if each is 6 inches by 6 inches?
6. In the room of Problem 5 there are five windows in the walls, each 3 feet wide and 6 feet high. How much wall surface is there, not counting the windows? (Does it matter where the windows are placed?)
7. In Problem 6 how many quarts of paint are necessary to paint these walls. A quart of paint will cover 132 square feet.
8. A trunk is 3 feet long, 18 inches wide, and 2 feet high. The edges are all covered with strips of brass. How much brass stripping is necessary? Express the answer in inches, in feet, and in yards.

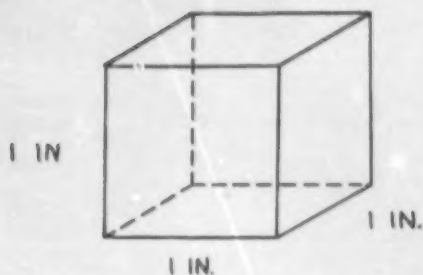
9. A cube is a rectangular prism of which all the edges are equal in length.
 - a. What is the shape of each face of a cube?
 - b. How many square inches of wood are needed to make a cubical box, including lid, with edges of 18 inches each?
 - c. Express your answer in square feet.
 10. Let l , w , h stand for the numbers of units in the length, width, and height of a rectangular prism. Write a number sentence telling how to find the number S of square units in the surface area.
-

12-5. Volume.

The term rectangular solid will be used to refer to the set of points consisting of a rectangular prism and its interior. The measurement of a rectangular solid is called volume.

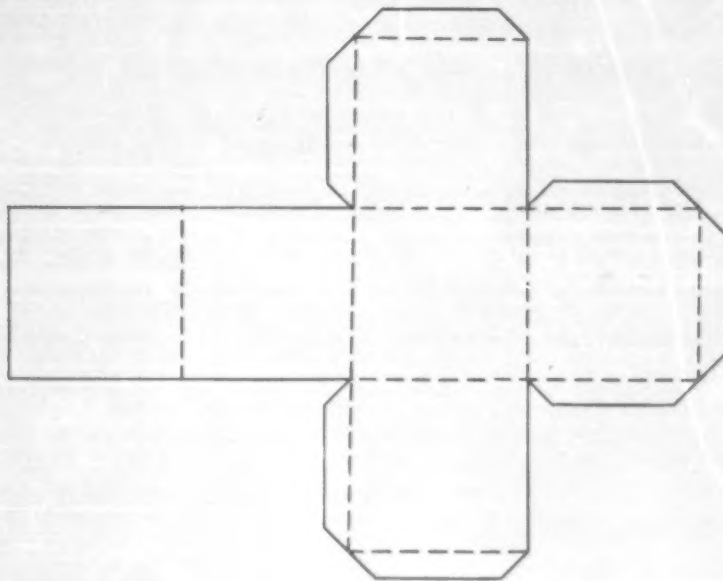
Do you recall that we were able to find the area of a rectangle just by working with its length and width? Do you agree, then, that it would also be useful if we had a way of finding the volume of a rectangular solid by working with its length, width, and height? First, it is necessary to agree on a unit of volume. The usual choice for a unit of volume is a cube. A cube is a rectangular prism for which all the edges have the same length. Would this have been your choice, or would you have chosen something else? What size cube would you suggest? (Remember that we chose a 1-inch square as our unit of area.)

The usual choice is a cube, each edge of which is a unit of length. In this case what can be said about the size of the faces? It is the neat relationship between the units of volume, area, and length which makes it easy for us to figure out the volumes. If we choose to measure lengths in inches then the unit of volume is a cube, each edge of which is 1 inch. The unit of measure of this cube is called a cubic inch.



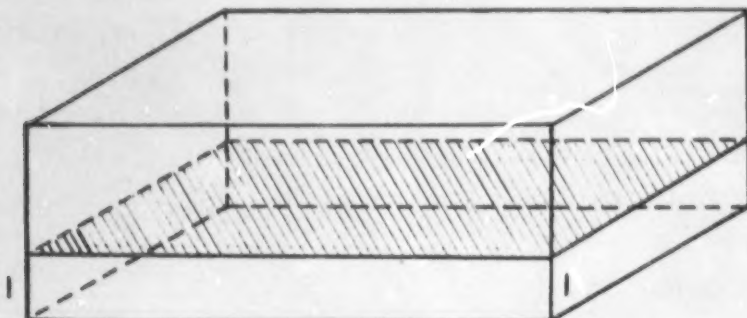
Class Exercises 12-5

1. Describe and name at least two other units of volume.
2. Make a model out of paper or light cardboard of a one-inch cube using the pattern below. The dotted lines show where to make your folds. The flaps are needed to paste the model together.



3. Obtain some small cubic blocks and think of the edge as a unit of length. Put these cubes together to form a rectangular solid 4 units by 3 units by 2 units. How many blocks did it take? (You and your classmates may put your model cubic inches together for this.)

4.
 - a. Use small cubic blocks to make a solid 3 units by 2 units by 2 units.
 - b. Make a rectangular solid with the length twice as long but the height and width the same. (6 by 2 by 2)
 - c. Make a solid with the width twice as long as the original but the measures the same. (3 by 4 by 2)
 - d. Make a solid with the height twice as long as the original but the other measures the same. (3 by 2 by 4)
 - e. Compare the number of cubes in parts b, c, and d with the number of cubes in part a.
 5.
 - a. Write a statement telling what happens to the volume of a rectangular solid if one of the measurements is doubled.
 - b. What is the ratio of the larger volume to the smaller?
 6.
 - a. Build a solid 6 units by 4 units by 2 units. Notice that both the length and width of the prism in 4(a) are doubled.
 - b. Find the ratio of the number of cubes in the new solid to the number in the original.
 - c. Would you get the same ratio if you doubled a different pair of measurements?
 7. Write a statement telling what happens to the volume of a rectangular solid if two of the measurements are doubled.
 8.
 - a. Double each measurement of the solid in Problem 4(a) and construct a rectangular prism. (6 units by 4 units by 4 units.)
 - b. Compare the number of cubes in this solid to the number in the original solid.
 9.
 - a. Write a statement telling what happens to the volume of a rectangular solid if all three measurements are doubled.
 - b. State the ratio of the larger volume to the smaller volume.
-

12-6. Volume of a Rectangular Solid.

We know how to find the area of any face of a rectangular prism. Let us imagine we have found the area of the face on which the solid is shown to be resting. Suppose the area of this bottom face (often called the base) is 12 square units. If the base consists of 12 unit squares, let us place a unit cube on each of these squares. These unit cubes fill up a layer one unit thick across the bottom of the solid. Since there are 12 such cubes, the volume of the layer is 12 cubic units. You saw examples of such layers in the problems in the last section. Since the top of the layer is just like the bottom, a second layer can be placed on the first. A third layer can also be placed on the second. Suppose the height of the solid is 3 units. Then it will be exactly filled with 3 layers. The total number of cubes may be found by multiplying 12 by 3 since we have 3 layers with 12 cubes in each layer. We need 36 cubes to fill this solid. Since each cube measures 1 cubic unit, the volume of this solid is 36 cubic units. Let us write this statement in symbols. Suppose we let the letter V represent the units of volume, then

$$V = 3 \times 12$$

$$V = 36.$$

The volume is 36 cubic units.

If the height is not a counting number, such as $2\frac{1}{2}$ units, then two layers will not fill the solid, but three layers will be too much. We can slice the third layer horizontally and

use only half of this layer. The volume of the solid, then, is $(2\frac{1}{2})(12)$. In symbols we have

$$V = (2\frac{1}{2})(12)$$

$$V = 30$$

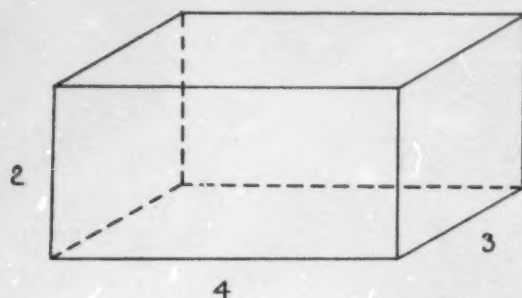
The volume is 30 cubic units.

Exercises 12-6a

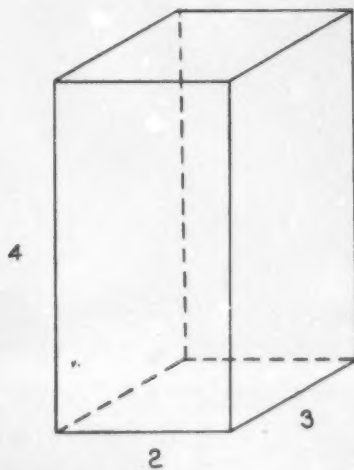
1. Find the volume of a closet $8\frac{1}{2}$ feet high if the area of the floor is 10 square feet.
 2. The base of a child's sandbox is a rectangle, and the area is 24 square feet. If the box is 10 inches deep, find its volume.
 3.
 - a. Write a statement in words telling how to find the number of cubic units in a rectangular solid if you know the number of square units in the base and the number of units in the height.
 - b. Suppose we let the letter B represent the number of square units in the base, h represent the number of units in the height, and V represent the number of units in the volume. Write the statement in part (a) in symbols.
 4. How deep should the sandbox of Problem 2 be made if the box is to hold 48 cubic feet?
 5. A man is making a wooden box to hold 260 cubic feet of sand. In order to fit into a certain space the box must be 13 feet long. How many square feet of area must be in the end of the box?
 6. Health regulations in a certain school district say that the school rooms must contain 50 cubic feet of air for each child. If a room has a floor area of 160 square feet and is 10 feet high, can the principal legally put 30 children in it? What is the greatest number of children who may be legally assigned to this room?
-

Look at the work of the last problems. Do you know in any of the problems the exact shape of the base? Is it necessary to know it? What must you know in order to find the volume of a rectangular solid? This procedure of finding the volume of a solid from the area of the base and height, without needing to know the exact shape of the base, will be used again when you consider other prisms and cylinders.

If we know all the edges of a rectangular solid, we know how to find the area of the base. Then we can find the volume.



In a rectangular solid 4 units by 3 units by 2 units, we probably would think of the largest face as the base. The area of this face is 4×3 square units so the volume, by Problem 3 above, is $2 \times (4 \times 3)$ cubic units. Notice that the number 4×3 which is enclosed in parentheses is the number of square units of area in the base. If we stand the solid on end we can think of another face as being the base.



We now find that the area of the base is 2×3 square units and the volume is $4 \times (2 \times 3)$ cubic units. Resting the prism on its third face, we find that the volume is $3 \times (2 \times 4)$ cubic units. Since it is the same solid in different positions we seem to be claiming that

$$2 \times (4 \times 3) = 4 \times (2 \times 3) = 3 \times (2 \times 4)$$

Is this true? What property or properties of rational numbers does it illustrate?

Exercises 12-6b

1. The inside of a freezer measures 4 feet long, 3 feet wide, and 2 feet deep. Find the volume of the freezer.
2. Find the number of cubic inches in a tool chest which is 14 inches long, 10 inches wide, and 9 inches deep.
3. The bottom of a desk drawer can be covered with 2 pieces of typewriter paper laid end to end. (Typewriter paper is $8\frac{1}{2}$ inches by 11 inches). The drawer is 4 inches deep.
 - a. Find the area of the bottom of the drawer.
 - b. Find the volume of the drawer.
4. A woman has some blankets to store. The blankets just fit into a trunk 3 feet long, $1\frac{1}{2}$ feet wide, and 2 feet high. How many cubic feet of space are in the trunk?
5. A man plans to make a sidewalk 81 feet long, 4 feet wide and 3 inches thick.
 - a. How many cubic feet of concrete does he need?
 - b. Express your answer in cubic yards.
6. Write a statement in words telling how to find the number of cubic units in the volume of a rectangular solid if the number of units in the length, width, and height are known.

7. Let l , w , h stand for the number of units in the length, width and height of a rectangular solid. Let V represent the number of cubic units in the volume. Write a statement in symbols telling how to find the volume of this solid.
8. a. How many cubic inches are there in a cubic foot?
b. How many cubic feet are in a cubic yard?
9. a. What is the volume of a 3-inch cube (a cube each edge of which is 3 inches.)
b. Is this volume larger or smaller than 3 cubic inches? Be very careful not to confuse the volume of a 3-inch cube with a volume of 3 cubic inches.
10. A rectangular solid is 2 inches long and 1 inch wide. The volume is 1 cubic inch.
a. Make a sketch of this solid.
b. What is the height of the solid?
11. A classroom is shaped like a rectangular prism. The length is 30 feet, the width is 28 feet, and the height is 12 feet.
a. Make a sketch of the classroom.
b. Find the volume.
12. A chalk box is 6 inches long, 4 inches wide, and 5 inches high.
a. Make a sketch of the chalkbox.
b. Find the volume.
c. Suppose the length, width, and height were each doubled. Find the ratio of the volume of this box to the volume of the original box.
13. a. Find the volume of a cube each edge of which is $1\frac{1}{2}$ inches.
b. Find the area of one face of this cube.
c. Find the total area of the cube.
d. Make a sketch of this cube.
14. A swimming pool in the form of a rectangular solid is 32 feet long and 20 feet wide. The pool is filled with water to a depth of $5\frac{1}{2}$ feet. How many cubic feet of water does the pool contain?

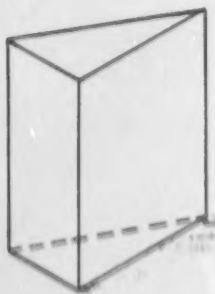
Since length, width, and height are always approximate, we should actually use the "approximately equal" sign $\approx$ in writing statements about measurements. In the exercise below use this sign where it is appropriate.

Exercises 12-6a

1. A stone block in the shape of a rectangular solid has a volume of $2\frac{1}{2}$ cubic yards. It weighs about 400 lbs. per cubic yard. What is its total weight?
2. An apartment house is built in the shape of a rectangular prism 210 feet long, 30 feet wide, and 30 feet high.
 - a. How many cubic feet of space is there in the building?
 - b. Express the volume in cubic yards.
3. A pirate's treasure chest was dug up and found to be filled with gold. The chest was a rectangular prism 2 feet 6 inches long, 18 inches wide, and 1 foot deep. A cubic foot of gold weighs 600 lbs. Could five men, each of whom can lift 400 lbs., lift the chest out of the hole?
4. Ed has a fish tank 24 inches long, 12 inches wide, and 14 inches high. He fills the tank to a height of 10 inches.
 - a. How many cubic inches of water are in the tank?
 - b. There are 231 cubic inches of water in one gallon. How many gallons of water are in the tank?
5. A steel bar is shaped like a rectangular prism. The bar is 18 inches long, $\frac{2}{3}$ inches wide and $\frac{2}{3}$ inches high. How many cubic inches of steel are in the bar?
6. An aquarium has the shape of a rectangular prism. The length is 40 inches, the width is 30 inches, and the height is 9 inches. The tank is filled with water. One cubic foot of water weighs about 62.5 lbs. How many pounds of water does the tank hold?

7. An electric fan is advertised as moving 3375 cubic feet of air per minute. How long will it take the fan to move the air in a room 27 ft. by 25 ft. by 10 ft.?
 8. The bricks which are used in many buildings are 8 inches by 4 inches by 2 inches. How many cubic inches of space does a brick contain?
 - *9. The length, width, and height of a rectangular prism are measured as $10\frac{1}{2}$ inches, $5\frac{1}{2}$ inches, and $3\frac{1}{2}$ inches. Find the volume from these measurements.
 - *10. A sandbox with a base 10 feet long and 9 feet wide is built in a park. A dump truck carrying 135 cubic feet of sand is emptied into the box. If the sand is leveled off, what is the depth?
-

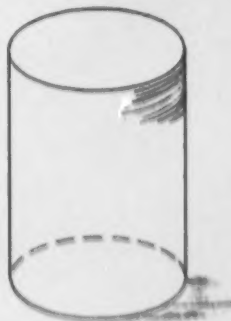
Probably the simplest possible solid is the rectangular prism. Of course, we shall learn later of other solids such as other prisms, cones, cylinders, and spheres.



Triangular Prism



Cone



Cylinder



Sphere

It is clearly a losing battle to try to consider all possible shapes. We can often obtain the results by adding or subtracting volumes we already know about. For example, suppose that in a room 15 feet long, 12 feet wide, and 8 feet high, a man builds

a closet in one corner. The closet runs to the ceiling and has a base 3 feet on a side, so that the floor plan looks like this:



How can we find the volume of the remaining space in the room? Let us try to devise a method.

1. Find the volume of the room before the closet was built.
2. Find the volume of the closet.
3. Now think of a way to find the remaining space after the closet was built.
4. Find the volume of this space.

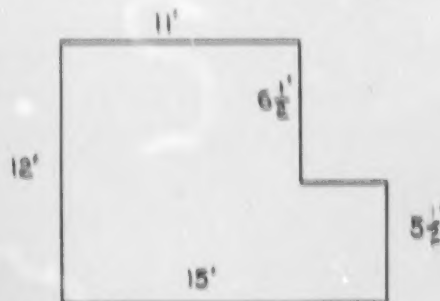
Suppose we see if all our answers agree. The volume of the entire room should be 1440 cubic feet. The volume of the closet should be 72 cubic feet. To find the volume of the remaining space we simply subtract these two volumes. Therefore

$$1440 - 72 = 1368$$

The volume of the remaining space is 1368 cubic feet.

Exercises 12-6d

1. The floor plan of a room is as shown below.

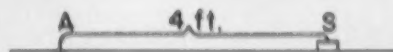


How many square feet of wall-to-wall carpeting would be necessary for the floor? What is the volume of the room if it is 9 feet high?

2. A pantry, the floor of which is 6 ft. by 10 ft., is 9 ft. high. It contains a deep freeze which is 2 ft. by 3 ft. by 7 ft. How many cubic feet of space are left in the room? Express the answer also in cubic yards.
3. BRAINBUSTER. Two boys were much interested in making flying models of airplanes. They made one model which flew very well, and decided they wanted a larger model just like it. They proceeded to build one by doubling all the dimensions (that is, twice the wingspread, twice the length, twice the height, etc.) and put in a motor with twice the power. To their disappointment, they found it would not fly at all. What was the trouble?

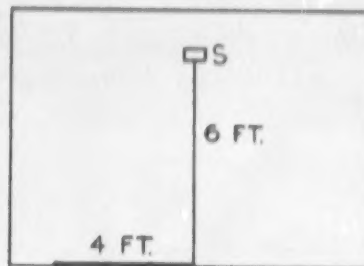
Dimension.

Consider two flies sitting side by side at a point A by the baseboard of a room. One of them is trying to direct the other to a lump of sugar which is also by the baseboard. What directions does he need to give?



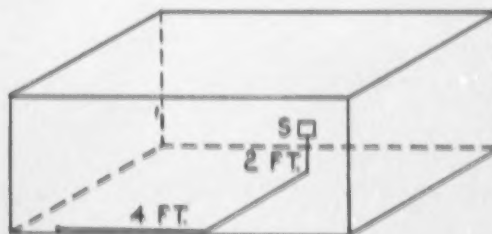
All he needs to say is "Follow the baseboard this way for four feet--you can't miss it!" The complete description of where the sugar is located by the baseboard can be given by one number. For this reason the edge of the room is called one-dimensional. Of course the section of baseboard followed may be a single segment, or may turn one or more corners, so any segment or simple closed curve is one-dimensional.

If the lump of sugar S is somewhere out in the middle of the floor, this presents more of a problem to the fly. His friend cannot get there at all by following the baseboard. How can he give directions? One of the ways would be as shown below.



"Follow the baseboard for four feet. Then turn to the left so you are headed perpendicular to the baseboard and crawl for six feet." In this case when the lump of sugar was in the interior of the rectangle it was found convenient to use two numbers to describe its location. For this reason the set interior to a rectangle (or any simple closed curve) is called two-dimensional.

If the lump of sugar is not on the floor at all but is somewhere else in the room, for example, suspended from the ceiling by a string, the problem of direction is harder still.



The directions then might go like this, "Crawl along the baseboard for four feet, then along the floor perpendicular to the baseboard for six feet. You will then be directly under the sugar. To get to the sugar fly directly up for two feet." This time we have used three numbers to describe how to get to the point S , so the interior of the room (that is, the interior of a rectangular solid) is called three-dimensional.

On the basis of the above discussion, what dimension would you want to give a point?

The dimension of the set where the fly is shows about how much freedom of motion he has. If he must stay in the one-dimensional set consisting of the floor's edge, he can only move one way or the other along this edge. If he may go anywhere in the two-dimensional set inside the rectangular edge he can crawl all over the floor. While if he is merely confined to the three-dimensional set interior to the room, he can fly all over the room.

The term dimension is also used in connection with more complicated sets. We do not try to give an exact definition, since it is usually clear whether a set is similar to a line, or to a region of the plane, or to the interior of a solid.

Exercises 12-6e

Indicate for each of the following sets whether it is one-dimensional, two-dimensional, or three-dimensional;

- | | |
|---------------------------|------------------------------------|
| a. triangle | i. exterior of an angle |
| b. interior of a triangle | j. rectangular prism |
| c. circle | k. interior of a rectangular prism |
| d. interior of a circle | l. exterior of a rectangular prism |
| e. sphere | m. surface of a desk |
| f. interior of a sphere | n. interior of a desk drawer |
| g. angle | |
| h. interior of an angle | |

Other Units of Volume.

In our discussion so far we have used the units of volume related to linear measure, such as the cubic inch, cubic yard, or cubic mile. In practice we often use other units of volume. If you go to the grocery, you ask for milk, cream, or vinegar in quarts or pints rather than in cubic feet or cubic inches.

Similarly you may ask for a bushel of peaches. There are definite relationships between these various measures and the cubic foot or cubic inch.

There would be a great advantage in doing away with most of these extra units of volume. Indeed, this is one of the great advantages of the metric system, about which you will hear a great deal as time goes on. However, since these units are in everyday use, we should know their relationships, or at least where to look them up. Unfortunately in our English system we even use different units for measuring liquid and dry quantities. The quart measure most people use is actually the liquid quart, but there is a dry quart which is somewhat larger. For convenient reference there is a section with information about the various units and their relationships, which is placed at the end of this chapter.

Exercises 12-6f

1. Milk often comes in quart containers that measure 7 in. by 3 in. by $2\frac{3}{4}$ in.
 - a. How many cubic inches are in the carton?
 - b. Is this $\frac{1}{4}$ of a gallon?
 - c. What is sometimes done to this container to make it look larger?
2. A quart milk carton has these measures: $2\frac{3}{4}" \times 2\frac{3}{4}" \times 7\frac{1}{4}"$. If the carton were filled would you have more than a quart or less than a quart in it?
3. There is an old saying, "A pint's a pound, the world around." Give a reason why this is not necessarily true.

4. Berries are often sold in boxes that are labeled pints and quarts. A "quart" box measures $4\frac{3}{4}$ in. by $4\frac{3}{4}$ in. by $2\frac{7}{8}$ in.
 - a. How many cubic inches does it contain?
 - b. If a "dry" quart is $1\frac{1}{6}$ times the size of a liquid quart and a liquid quart contains $57\frac{3}{4}$ cu. in., how many cubic inches are there in a dry quart?
 - c. Does the "quart" box of Problem 4 hold one dry quart?
5. A pint berry box measures $3\frac{3}{4} \times 3\frac{3}{4} \times 2\frac{1}{2}$ inches.
 - a. How many cubic inches does this contain?
 - b. How many cubic inches should there be in a dry pint?
 - c. How much bigger or smaller is the box than it should be?
6. Is there any reason why a "dry" quart should be larger than a liquid quart?
7. A bushel of apples is priced at \$3.25; apples can also be bought for 9¢ per pound. If a bushel holds 48 pounds of apples, how much do you save by buying a bushel?
8. A bushel of potatoes weighs 60 pounds. Which is cheaper, a bushel that costs \$3.50 or 60 lbs. bought at 4 lbs. for 25¢?
9. Half-gallon milk cartons have a base of $3\frac{1}{2}$ in. $\times$ $3\frac{1}{2}$ in. How tall should the carton be? If h stands for the number of inches in the height, write a number sentence for this problem.
- *10. The numbers of inches in the edges of a certain rectangular prism are all counting numbers greater than 1. If the interior of the prism has a volume of one gallon, what are the measurements of the prism?

12-7. Weight and Time.

The measures in this chapter have dealt with space only. Lines, surfaces, solids, and angles have been measured. There are many things that are measured that have no connection with space. Temperature is one example; time is another. Everyone

is familiar with weight as a measured quantity. The electricity used in your homes is measured. Can you name other things that are measured?

It is interesting to note that many of these items are measured by a scale marked on a line. When we read a thermometer, we are really reading a number line. In order to read such measurements, we read the measure of a line. Are any measures read on a circular scale? Measurement involves much more than finding the length of line segments.

Two more measurements will be considered briefly, time and weight. As with all other kinds of measurement, the units used to measure these things must have the same nature as the thing measured. Time is measured by units of time and weight is measured by units of weight. The scientific concepts of weight and time are complicated and will be left for later work in science. We will be concerned only with the use of some common units of weight and time.

Weight.

Units of weight are used to describe how "heavy" a given volume of something is. It may seem funny to think of your body as a volume, but it does occupy some amount of space and it does have weight. A bushel of feathers does not have as much weight as a bushel of potatoes, even though they occupy the same amount of space. The units of weight in the English system of measures are the ounce, the pound, and the ton. You should be familiar with the relations among these units.

$$16 \text{ oz.} = 1 \text{ lb.}$$

$$2000 \text{ lb.} = 1 \text{ ton (T)}$$

Time.

The basic unit of small divisions of time is the hour. Smaller units are formed by dividing the hour into 60 equal parts, each a minute. Each minute is divided into 60 equal parts, each called a second. Larger periods of time are related to the movement of the sun and moon. You should know the

relations among the following units:

60 sec. = 1 min.

60 min. = 1 hr.

24 hr. = 1 day

7 da. = 1 wk.

30 da. $\approx$ 1 mo.

52 wk. $\approx$ 1 yr.

365 da. $\approx$ 1 yr.

12 mo. = 1 yr.

You will notice that the relations between day and month, week and year, and day and year are only approximate. This is true because the earth takes a little longer than 365 days to make a complete circuit around the sun.

Exercises 12-7

1. Sea View School starts at 8:50 A.M. and closes at 3:30 P.M.
 - a. How many hours and minutes are in a school day at Sea View?
 - b. How many minutes are there in the school day?
 - c. If there are 8 equal class periods (including one for lunch) how many minutes in length is each class?
 - d. If there are only 7 equal periods, how long is each period?
2. The Sea View School is open 186 days each year. How many hours are there in a school year?
3. How many days are there between April 25 and May 6? Do not count either of these days.
4. How much time do you have each day for recreation if you sleep 9 hours, spend 6 hours and 45 minutes in school, study 2 hours, use 1 hour and 15 minutes for eating, and help your mother 1 hour?

5. A rectangular aquarium is 21 in. long, 18 in. wide, and is filled with water to a depth of 16 in. What is the weight of the water if 1 cubic ft. weighs $62\frac{1}{2}$ lb.? (This will be easier if you express the dimensions in feet.)
6. The label on Brand A of tomato juice states that the weight of the contents is 1 lb. 12 oz. The label on Brand B states that the weight of the contents of the can is 30 oz. Which brand has more tomato juice? How much more?
7. In a camp 70 people are fed at each meal.
- If each person has a 6 oz. serving of tomato juice how many ounces are needed for the camp for one meal?
 - How many pounds is this?
 - If Brand A is used, how many cans will be needed?
 - How many cans of Brand B?
 - Brand A costs 42¢ per can. Brand B costs 44¢. Which brand costs less to serve to the camp for one meal? How much less?
8. A ton of coal occupies 35 cubic feet.
- How many tons can be stored in a bin 8 ft. long, 7 ft. wide, and 5 ft. deep?
 - How many pounds is this?
9. BRAINBUSTER. Which weighs more, a cubic foot of water, or a cubic foot of ice? Why?
-

12-8. Computations with Measured Quantities.

There are many times when it is necessary to add or subtract measurements. You have already met many problems of this kind. To find the perimeter of a figure you add the length of the segments which make up the figure. When a number

is associated with some measuring unit, it is sometimes called a "denominate number", but we shall not use this term in our work. Let us consider the ways of performing arithmetic operations on these measured quantities.

Addition is the first arithmetic operation to consider, and addition of measured quantities makes no sense except when we add numbers of the same kind.

You cannot add the number of students in your class to the amount of money they have with them and have it mean anything. It makes no sense to add feet and gallons, or even the number of feet in one length to the number of inches in another. When we add measured quantities we therefore add only numbers of units which are alike.

So far, when we have given different names to a measured quantity, we have expressed each number wholly in terms of a single unit. For example, a length may be described by the name $3\frac{1}{2}$ ft. or by the name 42 in. This, of course, can always be done, but it is often more convenient not to do so. We are more likely to speak of a length of 2 ft. 5 in. than $2\frac{5}{12}$ ft, or 29 in. though all of these are correct names for the length. Similarly, the time of a train trip would more often be described as 2 hours, 17 min. rather than by the names 137 min. or $2\frac{17}{60}$ hrs. which are also correct. We shall here consider the composite names, though we know that we can always avoid them if we wish.

When you add measured quantities, remember that you will add only units which are alike. Write units which are the same in a column. When you add 3 yd. 2 ft. 7 in. to 5 yd. 2 ft. 10 in., you must make three separate additions as shown below.

3	yd.	2	ft.	7	in.
5	yd.	2	ft.	10	in.
8	yd.	4	ft.	17	in.

Notice that we added only the units which were the same. Can you write this answer in a better way? It is like having a pocket full of pennies. It would be much more convenient if they were changed to dimes or quarters or dollars. When you have enough small units to make one or more of the next larger unit, change the form of the answer. Some people call this "putting the answer in the best form" while others say "simplify" to mean the same thing. If we write 8 yd. 4 ft. 17 in. in the best form it becomes 9 yd. 2 ft. 5 in.

Exercises 12-8

Add and express the result in the best form:

$$\begin{array}{r} 1. \quad 3 \text{ hr. } 18 \text{ min.} \\ \quad 6 \text{ hr. } 52 \text{ min.} \\ \hline \end{array}$$

$$\begin{array}{r} 4. \quad 15 \text{ hr. } 10 \text{ min. } 50 \text{ sec.} \\ \quad 12 \text{ hr. } 55 \text{ min. } 18 \text{ sec.} \\ \hline \end{array}$$

$$\begin{array}{r} 2. \quad 5 \text{ ft. } 10 \text{ in.} \\ \quad \quad 5 \text{ in.} \\ \quad 1 \text{ ft. } 8 \text{ in.} \\ \hline \end{array}$$

$$\begin{array}{r} 5. \quad 3 \text{ sq. yd. } 5 \text{ sq. ft.} \\ \quad 2 \text{ sq. yd. } 8 \text{ sq. ft.} \\ \quad 6 \text{ sq. yd. } 7 \text{ sq. ft.} \\ \hline \end{array}$$

$$\begin{array}{r} 3. \quad 3 \text{ gal. } 3 \text{ qt.} \\ \quad 5 \text{ gal. } 2 \text{ qt.} \\ \quad 6 \text{ gal. } 1 \text{ qt.} \\ \hline \end{array}$$

$$\begin{array}{r} 6. \quad 25 \text{ cu. ft. } 500 \text{ cu. in.} \\ \quad 19 \text{ cu. ft. } 767 \text{ cu. in.} \\ \quad 12 \text{ cu. ft. } 1268 \text{ cu. in.} \\ \hline \end{array}$$

Subtraction, like addition, can be performed only when the units are the same, or can be changed so that they are the same. However, we sometimes find another difficulty in subtraction. Try this:

$$\begin{array}{r} \text{Subtract:} \quad 8 \text{ ft. } 4 \text{ in.} \\ \quad \quad \quad 2 \text{ ft. } 9 \text{ in.} \\ \hline \end{array}$$

How can you take 9 inches from 4 inches? We now use another important principle that you have used before. It is the reverse of the last step in addition. Before we subtract, we change the form of the minuend without changing its value. The work is shown here.

$$\begin{array}{r} 8 \text{ ft. } 4 \text{ in.} = 7 \text{ ft. } 16 \text{ in.} \\ \quad \quad \quad 2 \text{ ft. } 9 \text{ in.} \\ \hline \quad \quad \quad 5 \text{ ft. } 7 \text{ in.} \end{array}$$

Tell how each of the new numbers was obtained. When more of the smaller units are needed, convert one of the next larger units into the smaller ones.

Subtract:

- | | |
|--|---|
| <p>7. $\begin{array}{r} 5 \text{ yd. } 2 \text{ ft.} \\ \underline{3 \text{ yd.}} \end{array}$</p> <p>8. $\begin{array}{r} 6 \text{ min. } 10 \text{ sec.} \\ \underline{4 \text{ min. } 35 \text{ sec.}} \end{array}$</p> <p>9. $\begin{array}{r} 6 \text{ gal. } 1 \text{ qt.} \\ \underline{3 \text{ gal. } 3 \text{ qt.}} \end{array}$</p> <p>10 $\begin{array}{r} 3 \text{ ft. } 3 \text{ in.} \\ \underline{2 \text{ ft. } 8 \text{ in.}} \end{array}$</p> | <p>11. $\begin{array}{r} 4 \text{ lb. } 3 \text{ oz.} \\ \underline{12 \text{ oz.}} \end{array}$</p> <p>12. $\begin{array}{r} 8 \text{ cu. yd. } 6 \text{ cu. ft.} \\ \underline{2 \text{ cu. yd. } 5 \text{ cu. ft.}} \end{array}$</p> <p>13. a. Subtract: $\begin{array}{r} 41 \text{ } 4 \text{ ft. } 1 \text{ in.} \\ \underline{19 \text{ } 1 \text{ ft. } 9 \text{ in.}} \end{array}$</p> <p>b. In both cases did you have to exchange one larger unit for its equal in smaller units? Are the numerals in the answers alike? Explain.</p> |
|--|---|

Sometimes it is necessary to multiply measured quantities by an abstract number. Here is an example of a problem in which it is necessary. 2 feet 8 inches of webbing are needed to make one belt for students on the safety patrol. If Lincoln School has 15 members on the safety patrol, how many yards of webbing are needed? We could put 2 ft. 8 in. down 15 times and add but it is much quicker to use the short-cut, multiplication. We multiply each unit separately and change the answer to the best form, or the form that the problem asks for.

$\begin{array}{r} 2 \text{ ft. } 8 \text{ in.} \\ \underline{15} \end{array}$

$30 \text{ ft. } 120 \text{ in.} = 40 \text{ ft.} = 13\frac{1}{3} \text{ yd.}$ 40 ft. is the best form, but $13\frac{1}{3} \text{ yd.}$ is the form that the problem asks for.

Multiply:

- | | |
|---|---|
| <p>14. $\begin{array}{r} 6 \text{ hr. } 18 \text{ min.} \\ \underline{5} \end{array}$</p> <p>15. $\begin{array}{r} 2 \text{ ft. } 27 \text{ in.} \\ \underline{19} \end{array}$</p> | <p>16. $\begin{array}{r} 3 \text{ qt. } 1 \text{ pt.} \\ \underline{8} \end{array}$</p> <p>17. $\begin{array}{r} 5 \text{ hr. } 18 \text{ min.} \\ \underline{7} \end{array}$</p> |
|---|---|

$$18. \quad \frac{2 \text{ T } 1689 \text{ lb.}}{36}$$

$$19. \quad \frac{8 \text{ sq. ft. } 127 \text{ sq. in.}}{82}$$

When it is necessary to divide a measured quantity by a number, it is sometimes easiest to change all measurements to the smallest size unit, before doing the division. After division simplify the answer.

Here is an example. Divide 3 yd. 2 ft. 8 in. into 8 equal parts.

$$8 \overline{) 3 \text{ yd. } 2 \text{ ft. } 8 \text{ in.}}$$

$$3 \text{ yd.} = 108 \text{ in.}$$

$$2 \text{ ft.} = 24 \text{ in.}$$

$$\begin{array}{r} 8 \text{ in.} \\ \hline 140 \text{ in.} \end{array}$$

$$\begin{array}{r} 17\frac{1}{2} \text{ in.} \\ 8 \overline{) 140} \end{array}$$

$$17\frac{1}{2} \text{ in.} = 1 \text{ ft. } 5\frac{1}{2} \text{ in.}$$

Divide:

$$20. \quad 8 \overline{) 6 \text{ hr. } 16 \text{ min.}}$$

$$23. \quad 9 \overline{) 16 \text{ gal. } 8 \text{ qt.}}$$

$$21. \quad 10 \overline{) 23 \text{ yd. } 1 \text{ ft.}}$$

$$24. \quad 17 \overline{) 4 \text{ sq. ft. } 87 \text{ sq. in.}}$$

$$22. \quad 15 \overline{) 23 \text{ T. } 140 \text{ lb.}}$$

$$25. \quad 76 \overline{) 2 \text{ cu. ft. } 952 \text{ cu. in.}}$$

Changing all measurements into terms of the smallest unit is not necessary however. You may find it interesting to try Problems 21 or 22 above without it.

There is a very interesting relationship between the work in this section and the discussion of different number bases. For example, suppose we chose to use the number base twelve. Then the expression 2 ft. 7 in. could be written at once as 27_{twelve} in. or $\frac{27}{10}_{\text{twelve}}$ ft. In the base twelve how would

you express the number of square inches in a square foot? The number of cubic inches in a cubic foot?

26. Give at least one example of familiar units in measured quantities which would be nicely expressed in the following number bases:
- | | |
|---------------|-----------------|
| a. base two | c. base sixteen |
| b. base three | d. base sixty |
27. BRAINBUSTER. An apple weighs $\frac{3}{4}$ oz. and $\frac{3}{4}$ of its weight. How much does the whole apple weigh?
-

12-9. Summary.

1. A rectangle is a simple closed curve with four sides and a right angle in each corner. The lengths of the two intersecting sides of a rectangle are called the length and width of the rectangle.
2. The perimeter of a rectangle is the total distance around the rectangle.
3. If l and w are the number of linear units in the length and width of a rectangle, the number of linear units in its perimeter may be found from the number sentence
$$p = 2 \cdot (l + w)$$
4. The area of a rectangle means the area of the closed region determined by the rectangle.
5. The standard unit for measuring area is a square inch. A square inch is the area of a square the sides of which are 1 inch.

6. The number of square units of area of a rectangle is the product of the number of linear units in the length and width. This may be written as the number sentence

$$A = lw$$

where A represents the number of square units of area; l and w represent the number of linear units in the length and the width.

7. A rectangular prism is a figure like a chalk box, your textbook, or a cigar box.
Every rectangular prism has six rectangular sides which are called faces. The line segments determined by two intersecting faces is called an edge of the prism. Every rectangular prism has 12 edges. A point where three faces intersect is called a vertex. Every rectangular prism has 8 vertices.
8. The sum of the areas of all the faces of a rectangular prism is called the surface area of the prism.
9. A rectangular solid is a set of points consisting of a rectangular prism and its interior. The measurement of a rectangular solid is called volume.
10. The standard unit for measuring volume is a cubic inch. A cubic inch is the volume of a cube each side of which is 1 inch.
11. The number of cubic units of volume of a rectangular solid is the product of the number of square units in the area of the base and the number of linear units in the height. This may be written as the number sentence

$$V = Bh$$

where V represents the number of cubic units in the volume; B represents the number of square units in the area of the base; and h represents the number of linear units in the height. Since the base is a rectangle, this sentence may also be written

$$V = lwh$$

12. When a number is associated with some measuring unit, it is called a denominate number.

12-10. • Chapter Review.

Exercises 12-10

1. Harry's father built a fence around their lawn. The lawn is rectangular and measures 75 feet in length and 50 feet in width.
 - a. Find the perimeter of the lawn.
 - b. Find the area of the lawn.
2. a. Find the volume of a $2\frac{1}{2}$ -inch cube.
 b. Find the surface area of this cube.

In Exercises 3 and 4, add and then express the result in the best form.

$$\begin{array}{r} 3. \quad 5 \text{ yd. } 2 \text{ ft.} \\ \quad 4 \text{ yd. } 2 \text{ ft.} \\ \hline \end{array}$$

$$\begin{array}{r} 4. \quad 42 \text{ min. } 20 \text{ sec.} \\ \quad 51 \text{ min. } 34 \text{ sec.} \\ \hline \end{array}$$

In Exercises 5 and 6, multiply and then express the result in the best form.

$$\begin{array}{r} 5. \quad 2 \text{ ft. } 18 \text{ in.} \\ \quad \quad \quad 16 \\ \hline \end{array}$$

$$\begin{array}{r} 6. \quad 2 \text{ T } 325 \text{ lb.} \\ \quad \quad \quad 32 \\ \hline \end{array}$$

In Exercises 7 and 8, subtract.

$$\begin{array}{r} 7. \quad 9 \text{ gal. } 1 \text{ qt.} \\ \quad 4 \text{ gal. } 3 \text{ qt.} \\ \hline \end{array}$$

$$\begin{array}{r} 8. \quad 9 \text{ sq. yd. } 5 \text{ sq. ft.} \\ \quad 2 \text{ sq. yd. } 8 \text{ sq. ft.} \\ \hline \end{array}$$

9. a. A cube has ? faces.
 b. A cube has ? vertices.
 c. A cube has ? edges.
10. Which is larger: a 3-inch cube or 3 cubic inches? How much larger is it?

11. A man puts a tile floor in his recreation room. The floor is rectangular in shape. Each tile is a 6-inch square. Find the area of the floor if the man uses 1344 tiles to cover the entire floor.
 12. A rug is 9 feet by 12 feet and shaped like a rectangle. If the rug costs \$6.75 a square foot, find the total cost of the rug.
-

12-11. Cumulative Review.

Exercises 12-11

How Are You Doing?

1. List the elements of the set of all days of the week whose names begin with R.
2. Write three multiples of 24.
3. Answer true or false.
 - a. Zero is a number which is both even and odd.
 - b. Twenty-four is a multiple of 3.
 - c. The first prime number is 1.
4. Write the set of factors of 50.
5. Write the Roman numerals.
 - a. 23
 - b. 19
6. List the common factors of 20 and 28.
7. On a motor trip, Mr. Tobin averages 42.5 miles per hour. At this rate how far does Mr. Tobin travel in 8.5 hours?
8. Write in symbols.
 - a. 3 is greater than 0.
 - b. 15 is between 13 and 17.
9. A workman was paid \$270 for a job which required 50 hours of his time. At this rate, what should be paid for a 40 hour job?

10. State which of the following ratios are equal.

a. $\frac{34}{51}$, $\frac{2}{3}$

b. $\frac{10}{8}$, $\frac{95}{76}$

c. $\frac{6}{14}$, $\frac{45}{105}$

11. A cigar box is 10 inches long, 5 inches wide, and 3 inches deep.

- Find the surface area of the box.
- Find the volume of the box.

12. The length and width of a rectangle have been measured as $3\frac{1}{2}$ inches and $2\frac{3}{4}$ inches so that $l \approx 3\frac{1}{4}$ and $w \approx 2\frac{3}{4}$.

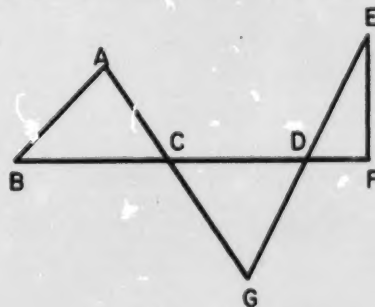
- Draw a rectangle with these dimensions.
- Find the approximate area.

13. In each case, describe the union of the two sets below.

- The set of prime numbers greater than 0 and less than 10 and the set of odd numbers greater than 0 and less than 10.
- The set of children in a family and the set of parents in the same family.

14. In the diagram on the right:

- What is $\overleftrightarrow{AG} \cap \overleftrightarrow{BF}$?
- What is $\overleftrightarrow{BF} \cup \overleftrightarrow{CD}$?
- What is $\overleftrightarrow{BD} \cap \overleftrightarrow{CF}$?
- What is $\overleftrightarrow{BD} \cup \overleftrightarrow{CF}$?
- What is $\overleftrightarrow{DE} \cap \overleftrightarrow{CB}$?



15. A segment appears to be $8\frac{1}{2}$ inches in length. Write the length of the segment to show precision of:

- $\frac{1}{2}$ inch.
- $\frac{1}{8}$ inch.
- $\frac{1}{16}$ inch.

TABLES FOR REFERENCE

English UnitsMetric Units

Measures of Length

12 in. = 1 ft.	10 millimeter (mm.) = 1 centimeter (cm.)
3 ft. = 1 yd.	100 cm. = 1 meter (M.)
$16\frac{1}{2}$ ft. = 1 rd.	1,000 mm. = 1 meter
320 rd. = 1 mi.	1,000 M. = 1 kilometer (km.)
5280 ft. = 1 mi.	

Measures of Surface

144 sq. in. = 1 sq. ft.	100 sq. mm. = 1 sq. cm.
9 sq. ft. = 1 sq. yd.	10,000 sq. cm. = 1 sq. M.
160 sq. rd. = 1 acre	1,000,000 sq. M. = 1 sq. km.
43,560 sq. ft. = 1 acre	
640 acres = 1 sq. mi.	

Measures of Volume

1728 cu. in. = 1 cu. ft.	1,000 cu. mm. = 1 cc.
27 cu. ft. = 1 cu. yd.	1,000,000 cc. = 1 cu. M.

Measures of Weight

16 oz. = 1 lb.	1,000 gram (g.) = 1 kilogram (kg.)
2000 lb. = 1 T.	1,000 kilogram = 1 metric ton

Liquid Measure

16 fluid oz. = 1 pt.	1,000 cc. = 1 liter (L.)
2 pt. = 1 qt.	
4 qt. = 1 gal.	

Dry Measures

2 pt. = 1 qt.	
8 qt. = 1 peck (pk.)	
4 pk. = 1 bu.	

Volumes are used for this.

Miscellaneous Measures

1 gal. = 231 cu. in.	
1 cu. ft. $\approx$ $7\frac{1}{2}$ gal.	
1 bu. $\approx$ 2150 cu. in.	
1 dry qt. $\approx$ $1\frac{1}{6}$ liquid qt.	

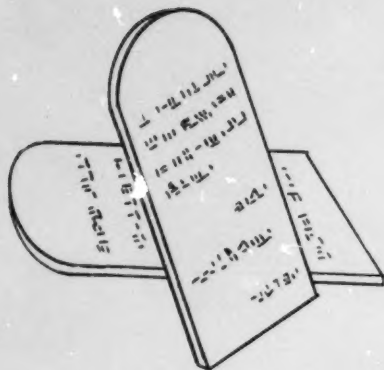
Metric and English Equivalents

1 in. = 2.54 cm.	1 cm. $\approx$ 0.4 in.
1 yd. $\approx$ 0.9 M.	1 M. $\approx$ 1.1 yd.
1 mi. $\approx$ 1.6 km.	1 km. $\approx$ 0.62 mi.
1 lb. $\approx$ 0.45 kg.	1 M. $\approx$ 39.37 in.
1 qt. $\approx$ 0.95 L.	1 kg. $\approx$ 2.2 lb.
	1 L. $\approx$ 1.05 qt.

Chapter 13

PARALLELS, POLYGONS, AND PRISMS

13-1. Two Lines in a Plane.



Early man was very much interested in the ideas of geometry. He also found many uses for them. Clay tablets, thousands of years old, show that ancient Babylonians knew how to find the area of a rectangle. This idea and other geometric ideas found their way to Egypt. The Egyptians added them to their own ideas and used their new knowledge to build great temples and pyramids 5000 years ago. As new ideas quickly grew out of the old ideas, it became important that they be put in order. Students could study them better that way. It was the Greeks who collected the scattered ideas and rules of geometry, wrote them down, and put them in order. To the collected ideas the Greeks added many new ones just as modern mathematicians are doing today.

Just as the Greeks did long ago, you will begin with a few of the simplest ideas of geometry and use these as stepping-stones to even more powerful ideas. Your study of geometry will give you some understanding of how the first parts of geometry grew out of the simplest beginnings.

For a simple beginning, suppose you think about two lines in space. What positions can they take in space? Can they be

parallel? Can they be skew? Can they intersect? Suppose they intersect. In the figure below, line l_1 and l_2 intersect in point A.



Perhaps the lines of the figure have drawn your attention to the angles in the figure. The angles are determined by the lines l_1 and l_2 . How many angles are determined by l_1 and l_2 ? Two angles determined by the lines are angle BAD and angle EAC. Can you name two more angles determined by line l_1 and line l_2 ?

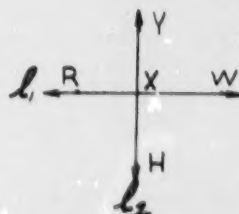
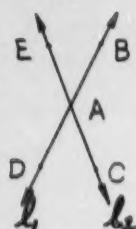
In the discussions about angles you will sometimes choose two angles of the four determined by the lines and use these two angles to show the truth of some geometric idea. It will be helpful to name certain pairs of angles which are used most often. One very important pair of angles includes angle CAD and angle DAB. If these two angles meet the following requirements, they are called adjacent angles:

1. Their interiors have no point in common.
2. They use the same point for a vertex.
3. They have a common side (ray).

Suppose you check these requirements for angles CAD and DAB. What is the name of the one point that both use for a vertex? Is $\overline{AD}$ a common side? Can any point on the C side of l_1 be a point on the B side of l_2 ? The answers to these questions indicate that angle CAD and angle DAB are adjacent angles.

Exercises 13-1a

1. Name four angles determined by l_1 and l_2 in each figure:
(Remember, the vertex letter is the middle letter.)



2. In which figure is a pair of adjacent angles marked with a star?



(a)

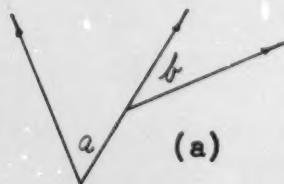


(b)

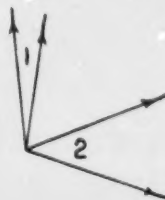


(c)

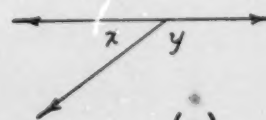
3. Which of these pairs of angles are not adjacent angles? Give a requirement that is not met by these pairs.



(a)

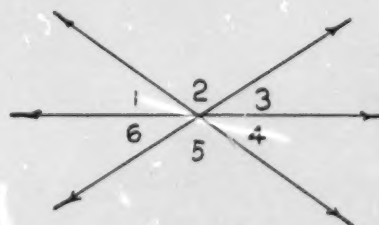


(b)



(c)

4. a. List six pairs of adjacent angles in the figure at the right.



- b. Draw two adjacent angles which have equal measures.

5. a. Draw two adjacent angles which have measures adding up to 180.
- b. Use only two lines to draw four different pairs of adjacent angles.
- c. Label points in the drawing for part (b) and name the pairs of adjacent angles. (Remember that the vertex letter of an angle goes between the other two.)

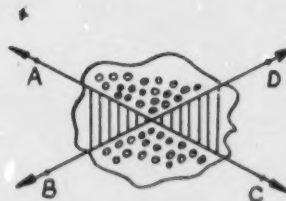
Perhaps the following figures look like a display of men's bow ties. Actually, they are to call to your attention pairs of vertical angles determined by two intersecting lines.



Notice that the angle pairs shown meet the following requirements:

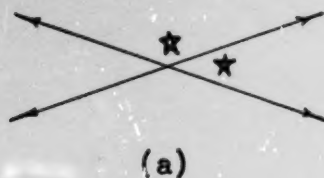
1. The same two lines determine each angle in a pair.
2. They are not adjacent angles. (They are non-adjacent angles.)

In the figure at the right, you will notice that when two lines intersect, two different "bow ties" or pairs of vertical angles are determined. One bow tie has a polka dot pattern and the other has a striped pattern. Name one pair of vertical angles in the drawing and then name the second pair of vertical angles. Look at the figures at the top of this page. Can you see the second pair of vertical angles in each figure?

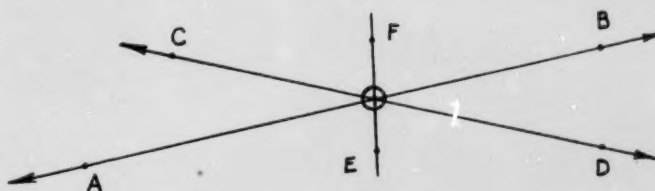


Exercises 13-1b

1. One of the pairs of angles in these figures that are marked with stars is a pair of vertical angles. In which figure is a pair of vertical angles marked?



2. Is it possible for vertical angles also to be adjacent angles? Why?
3. Name three pairs of vertical angles in this figure:



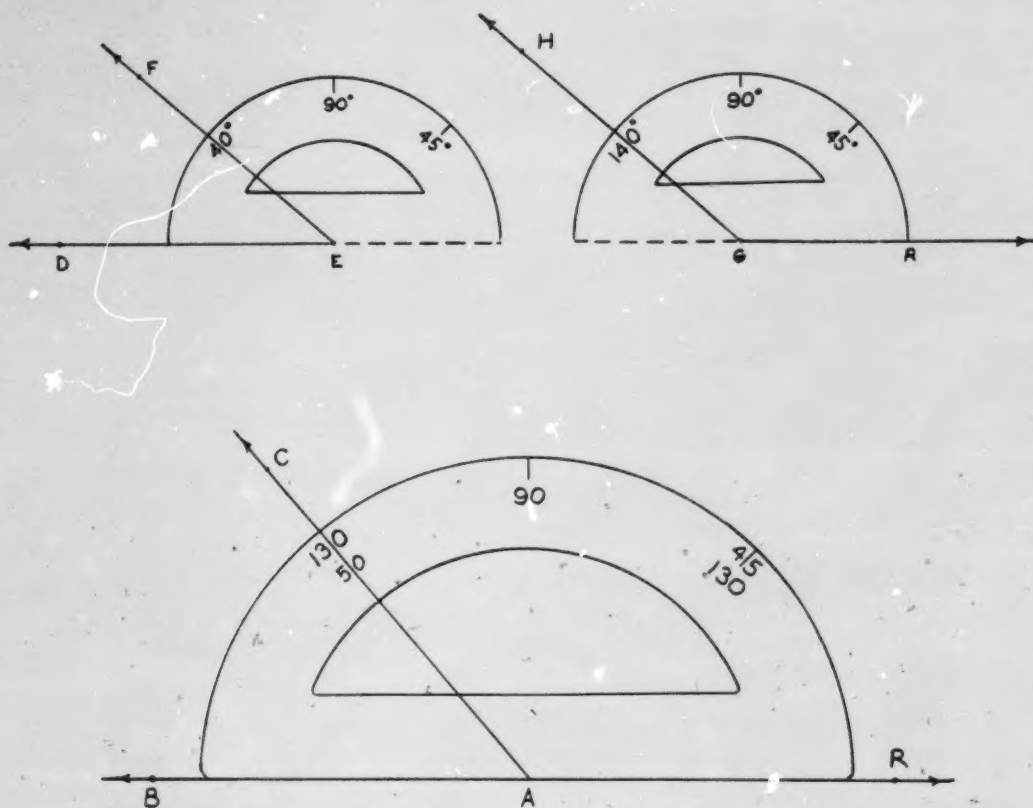
4. Name three more pairs of vertical angles in the figure for Exercise 3. (hint--add the measures of two angles to get a third.)
5. Draw two intersecting lines so that they determine vertical angles all of which are right angles.

Chapter 11 has a section that shows how to measure angles. The idea of angle measure is one on which many geometric ideas depend. Angle measure will be used so often in this chapter that it will be helpful to agree on symbols to express it. For the word "measure" the letter m is used. For "angle" you will use this symbol: $\angle$. With these symbols the words "measure of angle A" become simply: $m\angle A$.

Any angle can be used as a unit of measure but the degree is the one to be used in this chapter. In the future, unless some

other unit of measure is named, $m\angle A = 40$ means that angle A is a forty degree angle.

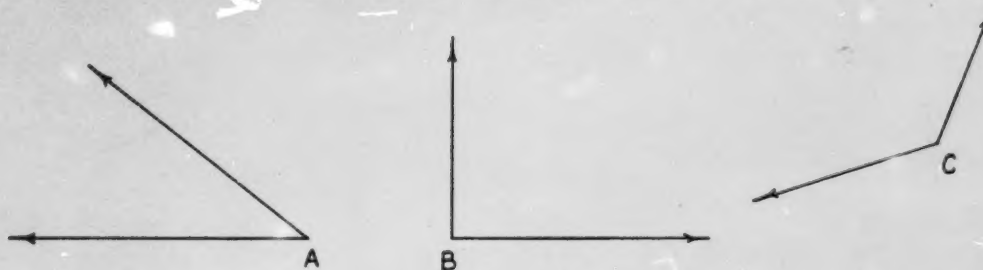
The drawings below show some protractors placed in position to measure several angles. The largest picture shows a protractor that has two numbers for each of its longer marks.



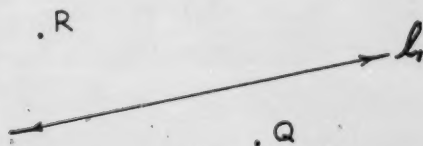
In the largest drawing above there are two angles which have measures totaling 180. The measure of angle BAC is 50 and the $m\angle CAR = 130$. The sum of 50 and 130 is 180. If the measures of two angles total 180, the angles are called supplementary angles. Thus, $\angle BAC$ and $\angle CAR$ are supplementary angles. This same idea is meant when we say " $\angle BAC$ is a supplement of $\angle CAR$."

Exercises 13-1c

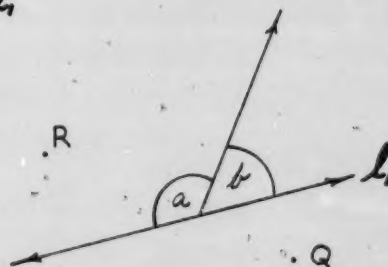
1. Measure these angles: (Use symbols to give your answers.)



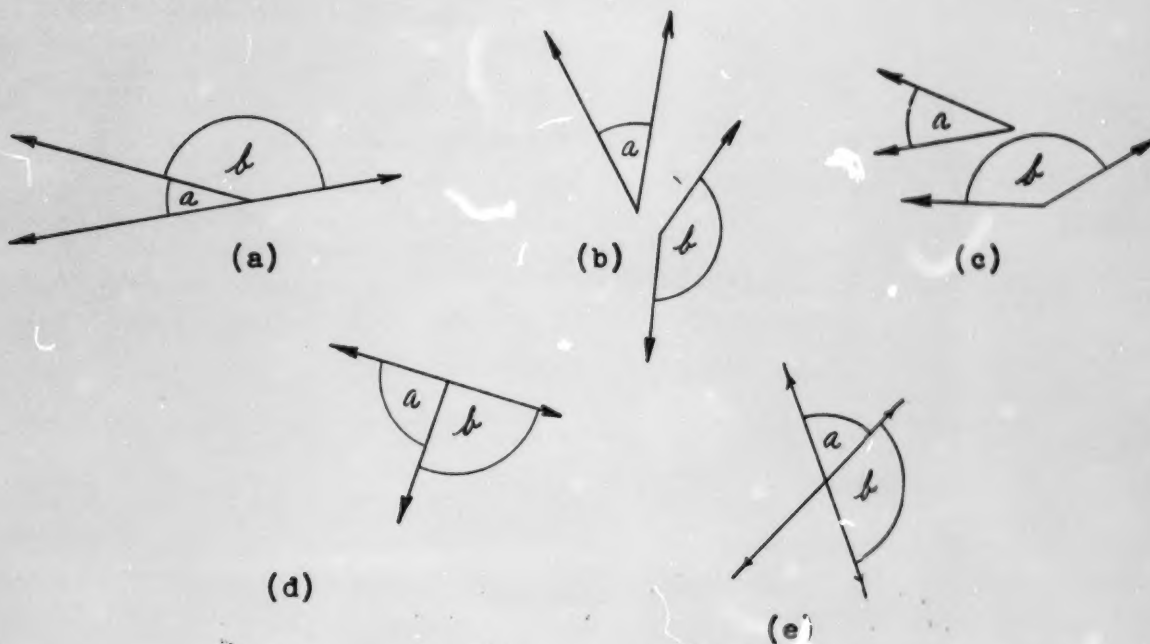
2. How many unit angles of one degree are contained in the half-plane containing point Q? How many unit angles of one degree are contained in the half-plane containing point R?



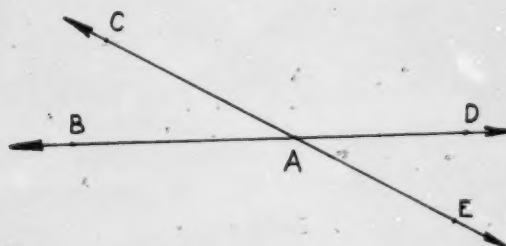
3. In the figure at the right, how many unit angles of one degree are in the interiors of $\angle a$ and $\angle b$ together? Are $\angle a$ and $\angle b$ supplementary? Is it always necessary to measure angles to find out if the angles are supplementary or not? Complete the following statement: If the union of the interiors of two adjacent angles and their common half-line is a half-plane, then the angles are _____ angles.



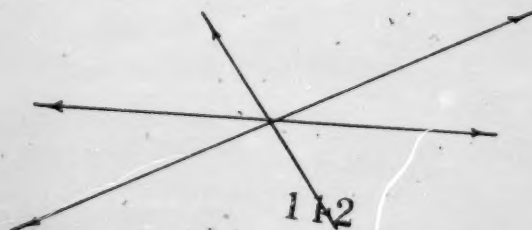
4. In which of these figures can you be quite sure that $\angle a$ and $\angle b$ are supplementary without actually measuring them? Why?



5. Name four pairs of supplementary angles in this figure:



6. Draw a pair of non-adjacent supplementary angles.
7. Draw a pair of supplementary angles which have equal measures.
8. How many pairs of supplementary angles can you count in this figure?



In the rest of this chapter there are a number of experiments. These experiments will help you discover new geometric ideas for yourself. In some ways, the plan of an experiment is like a map of a strange country. The plan gives you directions to follow. The map gives you roads to follow. But neither the map nor the plan warn you of many surprising things that lie ahead. These you must discover for yourself. Good hunting.

Experiment One

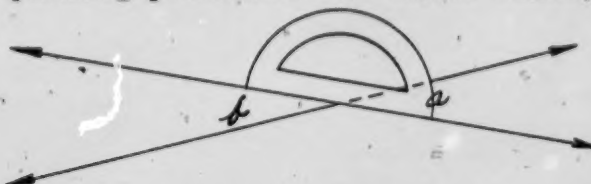
Discovering an Interesting Fact about Vertical Angles.

What is needed: Protractor, pencil, paper.

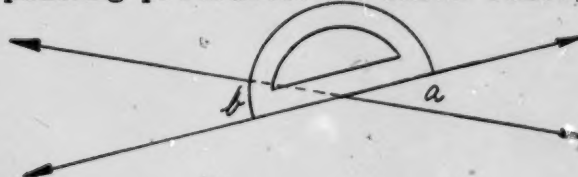
What to do: 1. Make an enlarged copy of the drawing below but make $m\angle a$ smaller or larger, whichever you please.



2. By placing protractor as shown below, measure $\angle a$.



3. By placing protractor as shown below, measure $\angle b$.



4. Record results in a copy of this table:

measure of $\angle a$	measure of $\angle b$

5. Do this experiment two more times for different measures of $\angle a$. Record the measures you have found in the table.

Questions: What seems to be true about the measures of two angles forming a pair of vertical angles? Check your answer by finding the measures of the other pair of vertical angles in each drawing.

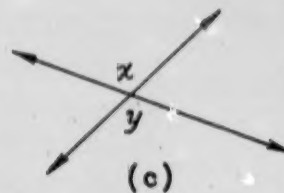
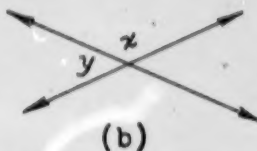
This seems to be true:

Property 1. If two lines intersect, the two angles in each pair of ? angles which are formed have ? measure.

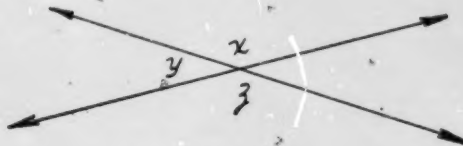
Exercises 13-1d

Do not use a protractor to find the answers for these exercises.

1. In which figure are $\angle x$ and $\angle y$ equal in measure? Give a reason why you chose this figure. ("They look equal" is not a good reason as you will see in a later section.)



2. The figure below is followed by some statements which are to be completed.



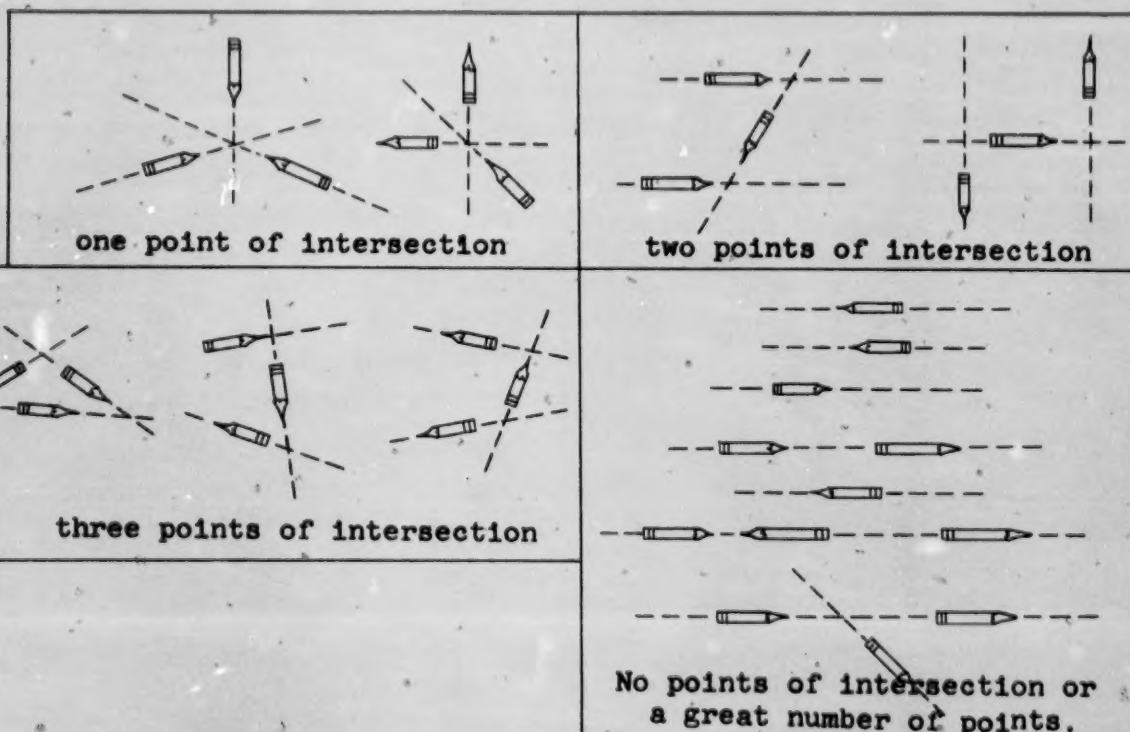
- $m\angle x + m\angle y = \underline{\quad ? \quad}$
- $m\angle z + m\angle y = \underline{\quad ? \quad}$
- If $m\angle y$ is known, how can you find $m\angle x$? How can you find the measure of angle z ?
- Using your answers to part (c) fill in these blanks:
 $m\angle x = \underline{\quad ? \quad} - \underline{\quad ? \quad}$
 $m\angle z = \underline{\quad ? \quad} - \underline{\quad ? \quad}$

- e. Notice that the right side of each equation represents the same number. Since this is true what can you say about the measure of $\angle x$ and the measure of $\angle z$?

13-2. Three Lines in a Plane.

The property of vertical angles which you have just found was known long ago. Thales, a Greek mathematician and teacher of about 600 B.C., is given credit for the idea. He traveled widely in Egypt and other countries of the ancient world and gathered together many of the geometric ideas known at that time. He and his students used these ideas to create and build new ideas as you are doing in this chapter.

Now that you know about two intersecting lines, the next step is to study three intersecting lines. Let three pencils represent three lines. Suppose they are tossed into the air and allowed to fall on a flat surface. No matter how many times this is done, the final positions in which the pencils come to rest may be described by the phrases below.

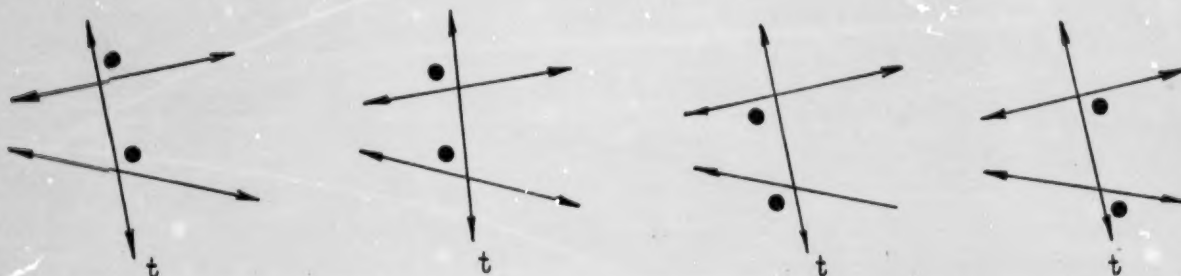


Which of these groups describe the most likely position in which three pencils would fall? Perhaps you could make up an experiment to test the answer to this question.

The group we shall examine first in our study of three lines in a plane is that which has three points of intersection.

Our next experiment will be about three lines belonging to the group that has three points of intersection. To get ready for this, we need a name for a certain pair of angles which are determined by lines in this group. The drawings below show four views of the same figure.

In each drawing the angles whose interiors contain a dot make up a pair for which a name is needed.



Two angles formed this way are called corresponding angles.

In the drawing at the right there are four pairs of corresponding angles. Angles which have interiors marked in the same way form a pair of corresponding angles.



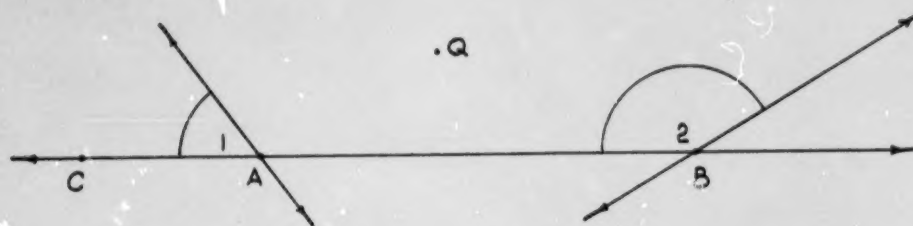
In the drawing at the left, the line t intersects two lines in separate points A and B . Line t is called a transversal of l_1 and l_2 .

Do you think there is a transversal in the drawing at the right? Is the intersection of these lines with t a set of separate points?



To remove any doubt as to whether a certain pair of angles are corresponding angles we shall use a list of requirements:

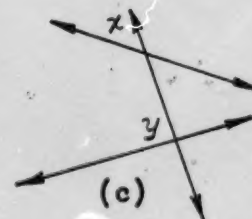
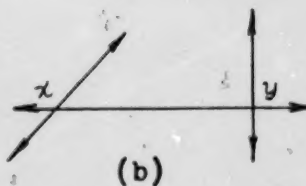
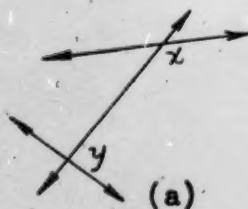
1. The interiors of the angles are on the same side of the transversal.
2. A ray of one angle is contained in a ray of the other angle.



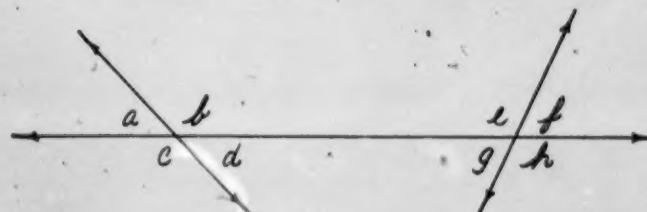
In the figure above, ray AC is contained in ray BC. Thus a ray of $\angle 1$ is contained in a ray of angle 2. Notice also that the interiors of $\angle 1$ and $\angle 2$ are both on the Q side of the transversal t. The requirements for corresponding angles are met by $\angle 1$ and $\angle 2$. Therefore $\angle 1$ and $\angle 2$ are corresponding angles.

Exercises 13-2a

1. In which of these figures is a pair of corresponding angles marked?



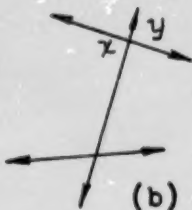
2. Name four pairs of corresponding angles in this figure:



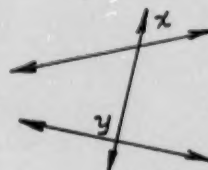
3. The angles marked in these figures are not corresponding angles. What requirement is not met in each pair?



(a)

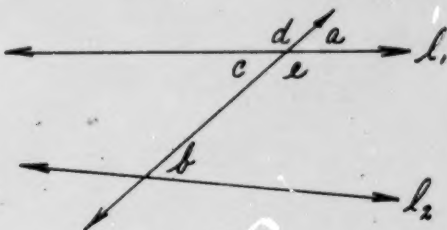


(b)



(c)

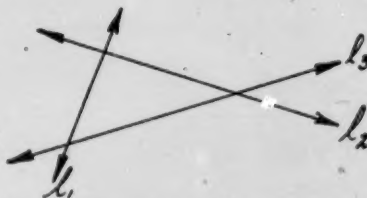
4. a. Can corresponding angles ever be adjacent angles as well? Why?
 b. Can corresponding angles ever be supplementary angles as well? Make a drawing that shown your answer is correct.
5. a. If $m\angle a = 80$ in the figure below is it possible to tell what the $m\angle b$ is without measuring? How?
 b. Is it possible to find the measures of the other angles marked in the figure without using a protractor? How?



6. Look again at the figure for Question 5. If you know the $m\angle b = m\angle c$, would you therefore know that $m\angle a = m\angle b$? Why?

7. In the figure at the right:

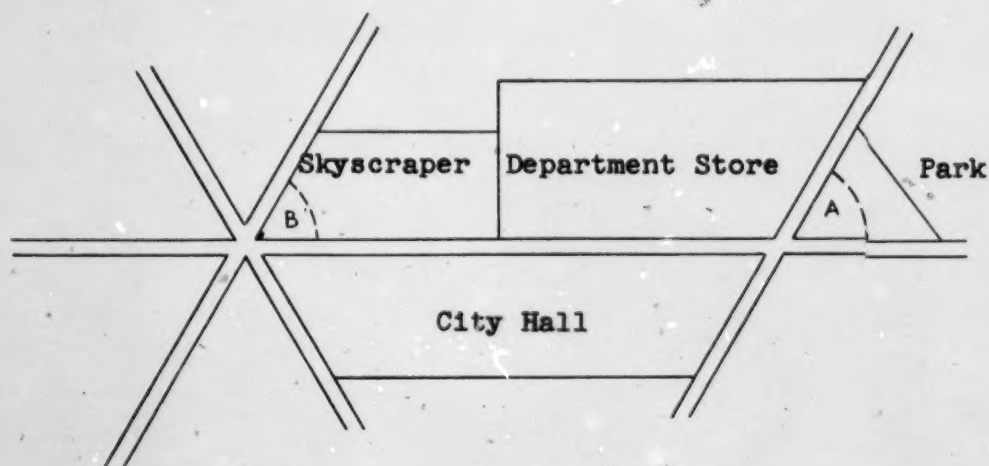
- a. Is l_1 a transversal? Why?
 b. Is l_2 a transversal? Why?
 c. Is l_3 a transversal? Why?



- d. How many pairs of vertical angles are there? How many pairs of adjacent angles? Supplementary angles? Corresponding angles?

13-3. Parallel Lines and Corresponding Angles.

Below is a section map of a large city where tall buildings hide one street from another. At point A, a policeman tells you that the street you are looking for is one block further at point B and is parallel to the street on which you are standing. On arriving at point B, however, you find two streets. It is impossible to see down the street where you met the policeman and see down the streets at B at the same time. The tall buildings block your view. How then, can you compare streets to see which ones are parallel? Luckily, you recall the size of angle A. Recalling this fact makes it possible for you to decide which street at B is parallel to the street at A. How is this done? What kind of angle pair is suggested by the angles marked at A and at B?

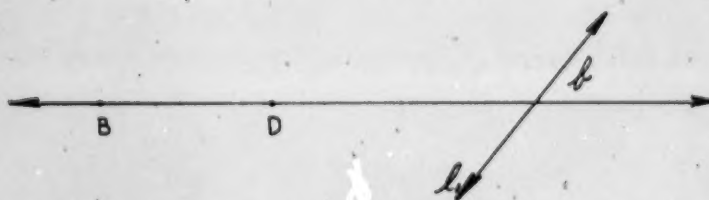


Experiment Two

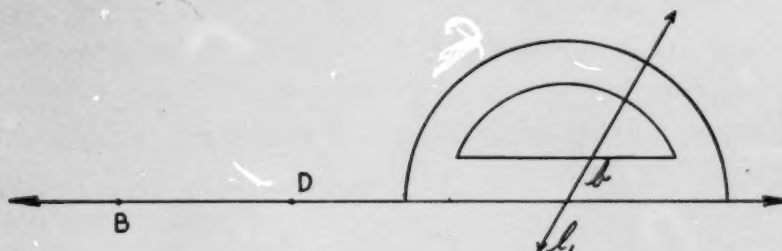
Discovering Some Properties of Corresponding Angles

What is needed: Protractor, pencil, paper.

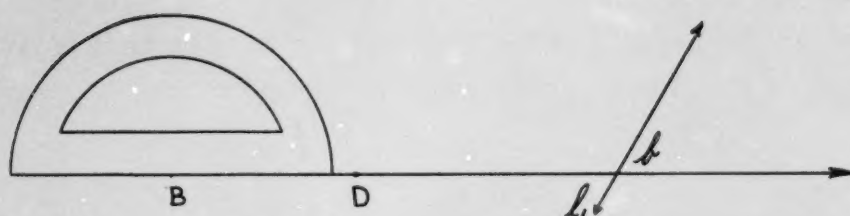
What to do: 1. Make an enlarged copy of the figure below but make angle b a different size.



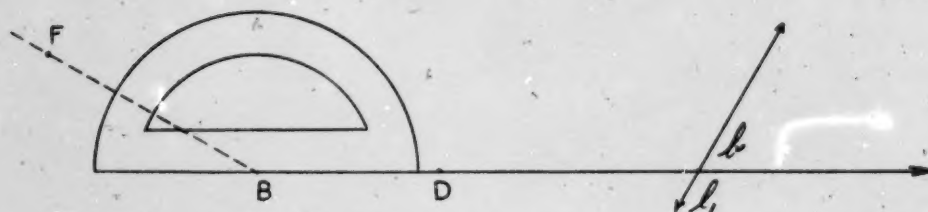
2. Measure $\angle b$. This drawing shows the protractor in place. Do not draw the protractor.



3. Place the center-point of the protractor on point B.



4. Locate point F so that $\angle DBF$ has a measure greater than that of $\angle b$.



5. Draw line BF. Record the angle measures in a copy of the table. In the third column answer "yes" or "no".

$m\angle b$	$m\angle DBF$	Does BF seem to intersect?

6. Repeat the experiment two times. The first time make the measure of $\angle DBF$ less than $m\angle b$. The second time make the measure of $\angle DBF$ equal to the measure of $\angle b$.

- Questions:
1. Are angle DEF and angle b corresponding angles? How can you tell?
 2. Did line BF ever intersect l_1 ? What were the measures of the corresponding angles when this happened?
 3. Will a pair of parallel lines result in all cases where two lines and a transversal form corresponding angles of the same measure?
 4. Is this idea helpful in finding the answer to the "parallel street" problem?

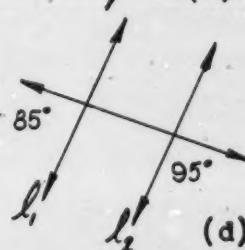
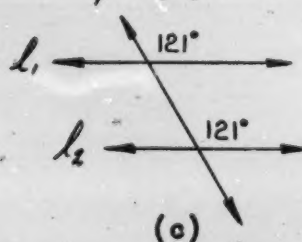
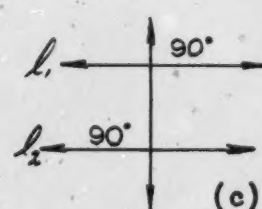
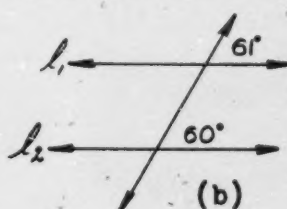
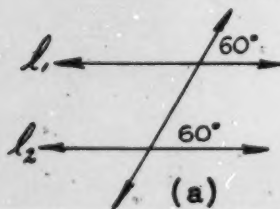
This seems to be true:

Property 2. When a transversal intersects two lines which lie in the same plane and a pair of corresponding angles have different ?, then the lines ?.

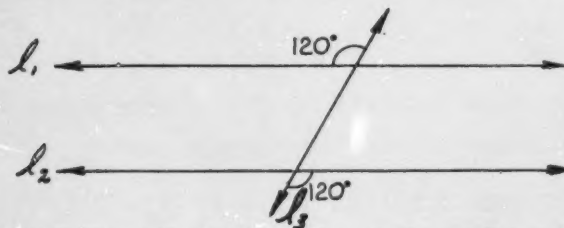
Property 2a. When a transversal intersects two lines in the same ? and a pair of corresponding angles have equal measures, then the lines are ?.

Exercises 13-3a

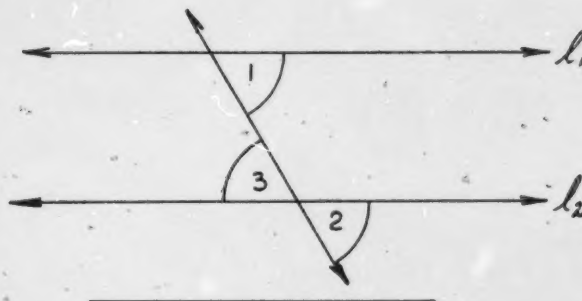
1. If the angles in the figures below have exactly the measures shown, in which of these figures do you think l_1 and l_2 will not meet? State a property as reason for your belief.



2. In the figures above, do l_1 and l_2 ever intersect? If so name the figure. State a property for your reason.
3. In the figure below will l_1 , l_2 and l_3 determine a triangle if the angle measures given are exact? Why?



4. a. If the measures of angle 1 and angle 2 are equal in the figure below, do you have reason to think that $m\angle 3$ also equals the $m\angle 1$? Why do you think these measures are equal?
- b. How many pairs of corresponding angles are in this figure?



13-4: Converse.

One of the many ways to get new ideas is to write a statement about an old idea and then change the words around according to the following plan:

First statement: If today is Monday then tomorrow is Tuesday.

Second statement: If tomorrow is Tuesday then today is Monday. Suppose we call the words underlined once the "if" clause and the words underlined twice the "then" clause. As the arrows show, the "if" clause and the "then" clause of the first statement have been exchanged to produce the second statement.

Let's try this way of getting ideas with another statement.

First statement: If Mary and Sue are sisters, then Mary and Sue are girls.

Second statement: If Mary and Sue are girls then Mary and Sue are sisters.

Interchanging the "if" clause and the "then" clause of a statement gives a new statement. This new statement is called the converse of the first statement. In each of the above samples, the second statement made is the converse of the first statement.

A converse of a geometric statement might suggest a geometric idea worth studying.

First statement: If $\angle a$ and $\angle b$ are vertical angles then $\angle a$ and $\angle b$ have the same measure.

Second statement: If $\angle a$ and $\angle b$ have the same measure then $\angle a$ and $\angle b$ are vertical angles.

In each of the examples on this page is the converse of the first statement true? Are all converses true? Are all converses false? We can see from these examples, that if a statement is true, a second statement obtained from it and called its converse may or may not be true. Could a false statement have a true converse?

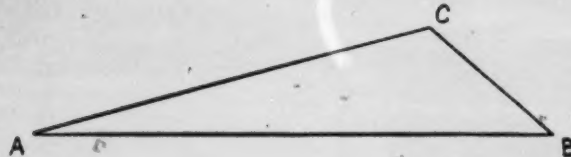
Exercises 13-4

1. Must any two angles which have the same measure be vertical angles? Make a drawing which shows that the geometric converse in the preceding paragraph is false.

2. For each of the following statements write "true" if the statement is always true; "false" if the statement is ever false.
 - a. If Blackie is a dog, then Blackie is a cocker spaniel.
 - b. If it is night, then we cannot see the sun.
 - c. If it is July 4th, then it is a holiday in the United States.
 - d. If Robert is the tallest boy in his school, Robert is the tallest boy in his class.
 - e. If an animal is a horse, the animal has four legs.
 - f. If an animal is a bear, the animal has thick fur.
 3. Write a converse for each statement in Problem 2 and tell whether the converse is true or false.
 4. Read the following statements. Write "true" if the statement is always true; "false" if the statement is sometimes false.
 - a. If a figure is a circle, then the figure is a simple closed curve.
 - b. If a figure is a simple closed curve composed of three line segments, then the figure is a triangle.
 - c. If two angles have equal measures, then they are right angles.
 - d. If two lines are parallel, then the lines have no point in common.
 - e. If two angles are supplementary, then they are adjacent.
 - f. If two adjacent angles are both right angles, they are supplementary.
 5. Write a converse for each statement in Problem 4 and tell whether the converse is true or false.
-

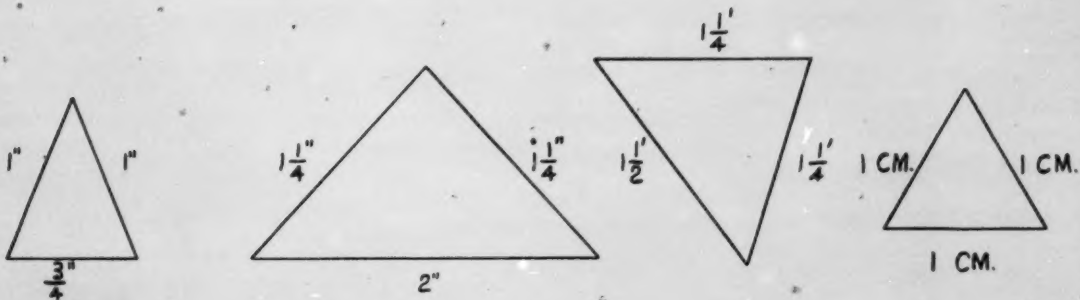
13-5. Triangles.

In the figure below there are three points not on the same line. Three segments join the points in pairs.

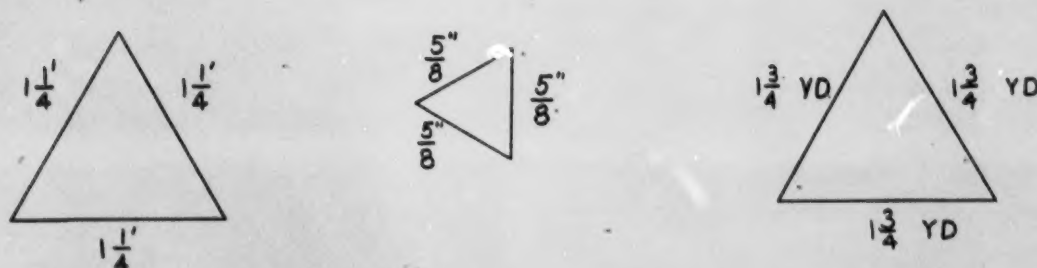


The figure is called a triangle. This triangle is the union of the segments AB, BC, and CA. The vertices of the triangle are the points A, B, and C.

If at least two sides of a triangle have the same length, the triangle is an isosceles triangle. Does a triangle having three sides equal in length meet the requirement of having at least two sides of the same length? Such a triangle, then, is also isosceles. All of the triangles below are isosceles triangles:

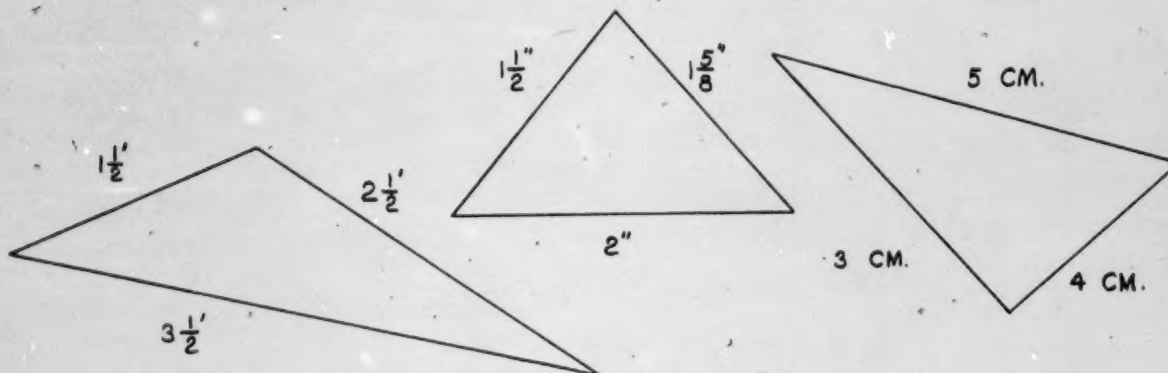


Are all of the triangles in the group below isosceles triangles?



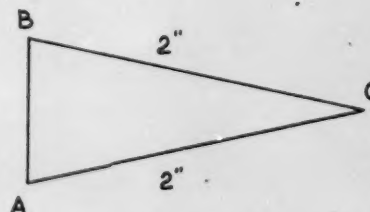
You may have noticed that each of the triangles in the figure above is an example of an isosceles triangle of a special kind. This special kind of isosceles triangle is called an equilateral triangle. All three sides of an equilateral triangle have the same length.

A triangle is a scalene triangle if no two of its sides have the same length. This is a group of scalene triangles:

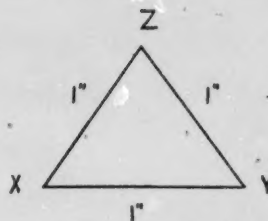


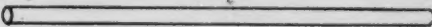
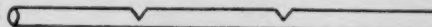
Exercises 13-5a

1. Draw a triangle in which each of two sides is one and one-half inches in length. What kind of triangle have you drawn?
2. Draw an isosceles triangle that has an angle of 90° .
3. One side of a triangle is two inches long, another is three inches long, and the third is one and a half inches long. What kind of triangle is it? Why?
4. a. What kind of triangle is shown here:
 b. If $\overline{AB}$ were made the same length as $\overline{AC}$, what kind of triangle would be formed? Why?



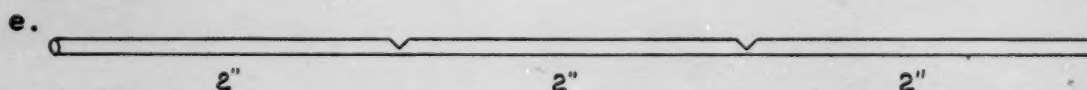
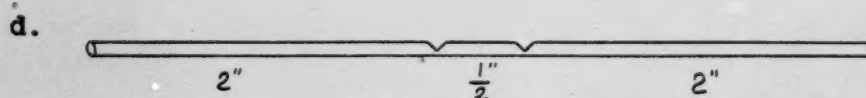
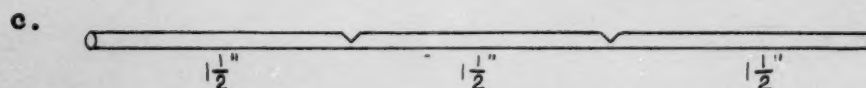
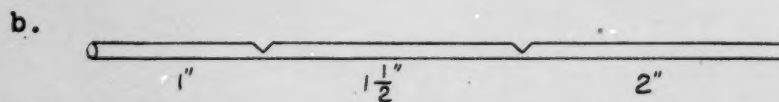
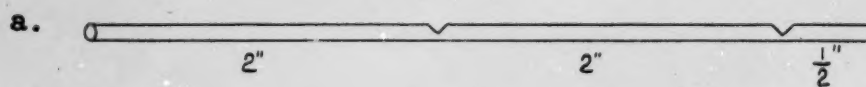
5. a. What kind of triangle is shown at the right?
- b. If $\overline{XY}$ were shortened by half an inch and triangle XYZ were redrawn, what kind of triangle would it be? Why?

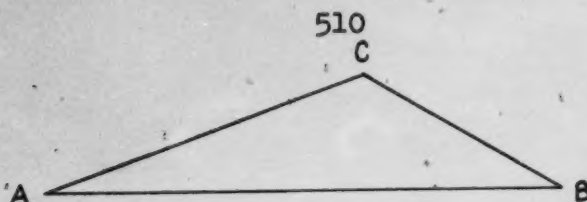


6. Here is a picture of a soda straw:  and here it is creased at two points:  ready to be folded so that the ends come together like this:



What kinds of triangles can be represented in this way by folding the following straws at the points indicated:





Place a fingertip on point A in the drawing above. In doing this you have to touch the mark representing $\overline{AC}$ and the mark representing $\overline{AB}$. The segment $\overline{BC}$ is not touched. Side BC is opposite angle A.

Place a fingertip on point B. This time the only mark which is not touched represents segment AC. Then, $\overline{AC}$ is opposite $\angle B$. We can also say that $\angle B$ is opposite $\overline{AC}$. Which segment is opposite $\angle C$? Which angle is opposite $\overline{AB}$?

Some triangles are so small that they can be completely covered by a fingertip and the above method of finding a side opposite an angle is not easy to use. Notice that only the endpoints of a side opposite an angle are contained in the angle. Using this idea we note in the above figure that points B and C are the only points of segment BC that are also points of angle A. Only the endpoints of segment BC are contained in angle A. Therefore segment BC is opposite angle A.

Experiment Three.

Two Important Facts about Angles in an Isosceles Triangle

What is needed: Protractor, ruler, pencil, drawing paper, tracing paper.

- What to do:
1. Draw three isosceles triangles that do not look alike.
 2. In each triangle find the intersection of the sides which have the same length. Label this point as point A.
 3. Label the other vertices of the triangle with the letters B and C.
 4. In one of the triangles make a tracing of $\angle B$ and then see if the tracing will fit $\angle C$. Do this again for each of the triangles that are left.

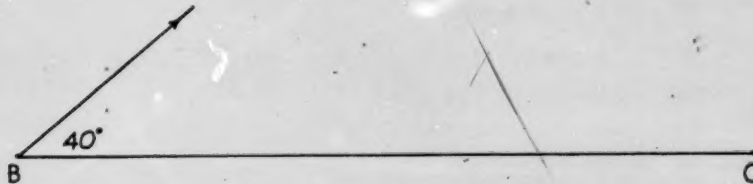
- Questions:
1. In each triangle name the sides equal in length.
 2. What angle is opposite side AB?
 3. What angle is opposite side AC?
 4. What do you notice about the measures of angles opposite equal sides in a triangle?

This seems to be true:

Property 3. If two sides of a triangle are ? in length, then the angles ? these sides have ? measures.

Part Two

- What to do: 1. Draw a segment BC about five inches long. With vertex B draw an angle of 40° .



2. Place the centerpoint of the protractor on point C and draw another 40° angle as shown;



3. Extend the rays until they intersect. Label the intersection F.

- Questions:
1. In making triangle CBF did you use sides of the same measure or angles of the same measure? Name the parts measured.
 2. Which side of the triangle is opposite angle B? Opposite angle C?

3. What fact about $\overline{FC}$ and $\overline{FB}$ would have to be true in order for triangle FCB to be an isosceles triangle? Measure $\overline{FC}$ and $\overline{FB}$ to see if triangle FCB is isosceles.

This seems to be true:

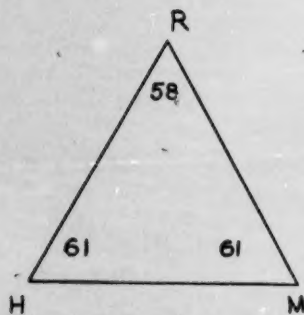
Property 3a. If two angles of a triangle are ?
in measure, then the sides ? these angles are
 ?.

Class Discussion Problems

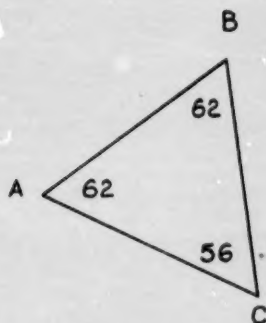
1. Could the three kinds of triangles be defined by using angle measures rather than side measures? How?
2. How could the converse of Property 3 be used to show that a triangle having three angles of the same measure must be an equilateral triangle?

Exercises 13-5b

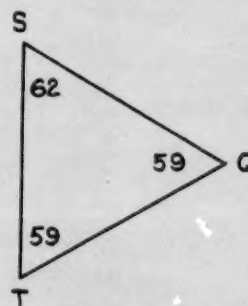
1. Name two segments of equal length in each of the following figures: (Do not measure the segments.)



(a)

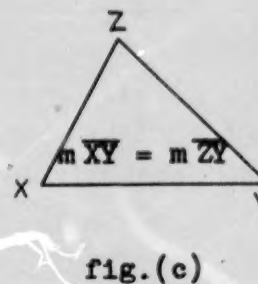
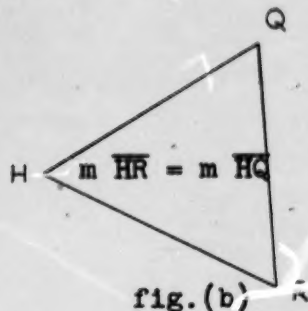
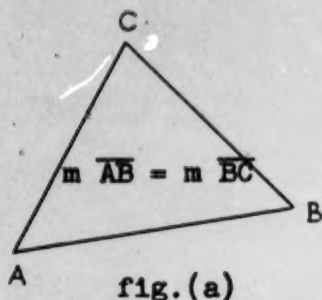


(b)

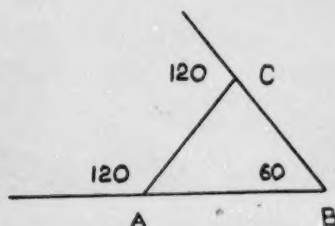
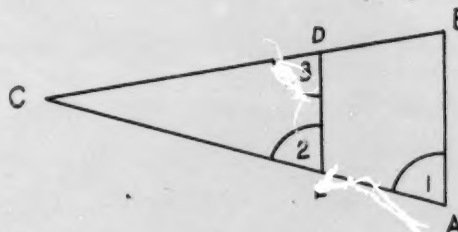


(c)

2. Name a pair of angles with equal measures in each of these figures:



3. Draw a triangle which has two angles whose measures are equal. What kind of triangle have you drawn? Name the equal sides.
4. Draw an equilateral triangle with 3" sides.
5. Are all equilateral triangles also isosceles? Explain your answer. Are all isosceles triangles also equilateral? Explain your answer.
6. In the figure at the right these two facts are known:
1. $\overline{AB}$ and $\overline{DE}$ are parallel.
 2. $m\angle DC = m\angle CE$.
- a. Why does $m\angle 1 = m\angle 2$?
 - b. Why does $m\angle 3 = m\angle 2$?
 - c. Make a true statement about the measures of $\angle 1$ and $\angle 3$.
7. If the angles of the following figure have the exact measures shown, what kind of triangle is triangle ABC? Why?

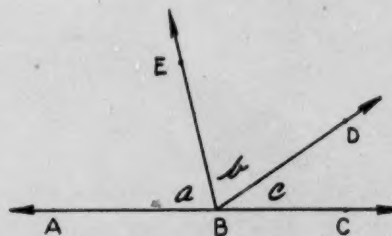


13-6. Angles of a Triangle.

In the section before this, we studied special properties for certain triangles. In this section we come to a property which is true for all triangles.

Recall that the sum of the measures of $\angle a$, $\angle b$, and $\angle c$ in the figure at the right is 180.

This is true because the line ℓ , determines a half-plane every point of which is in either the interior or a half-line of the labeled angles. Since 360 unit angles are contained in the complete plane, half of this number, or 180 unit angles, are contained in the half-plane above ℓ .

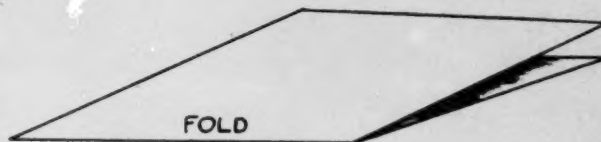


Experiment Four

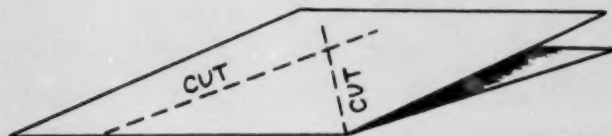
Ways to Find the Sum of Angle Measures in a Triangle

What is needed: Scissors, paper, pencil, protractor.

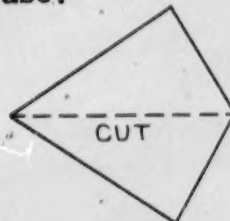
What to do: 1. Fold a piece of paper in half.



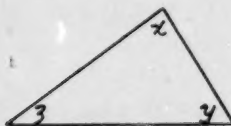
2. Draw two lines which intersect the folded edge as shown below. Cut both halves of the paper at once along these lines.



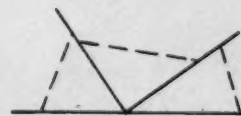
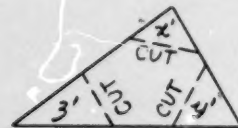
3. Unfold the cut out part and then cut this part along the crease.



4. You now have two triangular regions. Label the angles in each as shown. Notice that the angles in the triangle at the right have little dashes over them. The names of these angles are "x prime", "y prime", and "z prime".



5. Cut the corners off of the region with the "prime" marks as shown below. Fit the corners together as shown at the right.



- Questions:
- In the following questions the angle determined by the edges of corner piece y' will be called "angle y prime". The piece marked x' determines "angle x prime". The piece marked z' determines "angle z prime".
 - Does angle x' fit on angle x ?
Does angle y' fit on angle y ?
Does $\angle z'$ fit on $\angle z$?

3. Do you agree that the measure of angle x' is the same as the measure of angle x ?

Does $m\angle y' = m\angle y$?

Does $m\angle z' = m\angle z$?

Is the sum of the measures of $\angle x'$, $\angle y'$, and $\angle z'$ the same as the sum of the measures of $\angle x$, $\angle y$, and $\angle z$?

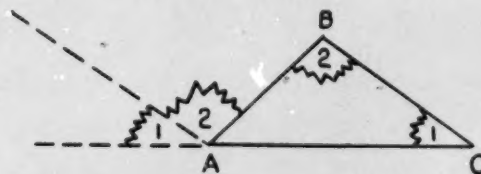
4. In Step 5 of the "What to do" section you fitted the corner pieces together in a certain way. What do you think is the sum of $m\angle x'$, $m\angle y'$, $m\angle z'$? What is the sum of the angles determined by the edges of the uncut region? Do your classmates agree with you about this?

What seems to be true:

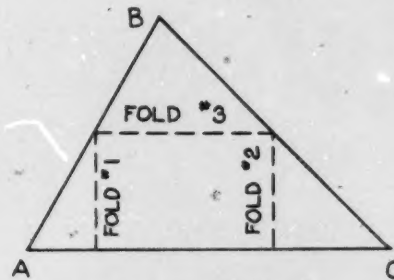
Property 4. The sum of the measures in degrees of the angles of any triangle is ?.

Exercises 13-6a

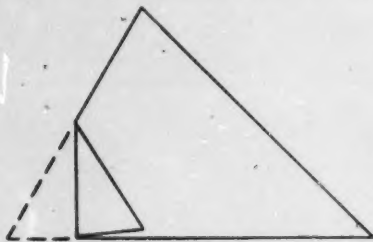
1. Draw a triangle, making each side from two to three inches in length. Cut out this triangular region. Tear off the corners and mount the whole figure on cardboard or a sheet of paper. As shown on the right, corners B and C are placed around the vertex A. What do you observe about angle 1, angle 2, and angle BAC in this new arrangement?



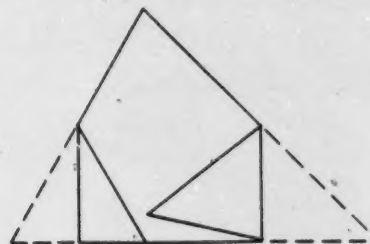
2. As the following drawings show, cut the interior of a triangle ABC out of paper and mark on AB a point midway between point A and point B. On BC mark a point midway between B and C. Fold the triangular region on the dotted lines.



Fold the triangular region according to the following steps:

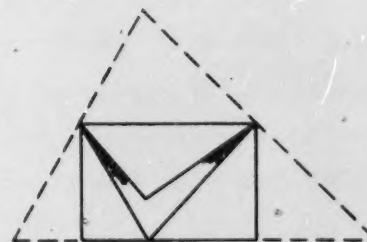


Step 1. (Fold 1 started)



Step 2. (Fold 1 completed
Fold 2 started.)

Notice the point where the vertices of all three angles of the triangle finally come to rest. Explain how this exercise seems to show that Property 4 is true.



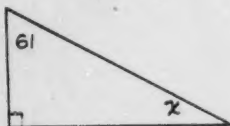
Step 3. (Folds 1 and 2 completed
Fold 3 started.)

Do not use a protractor for these exercises. Use Property 4.

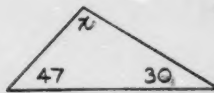
3. Find the measure of $\angle x$.



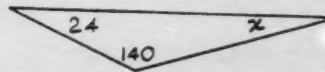
(a)



(b)

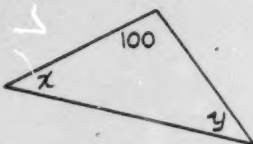


(c)

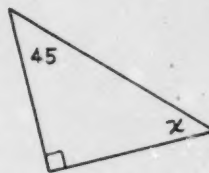


(d)

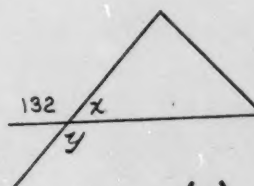
4. Find the measures of $\angle x$ and $\angle y$. Figures (a) and (e) are isosceles.



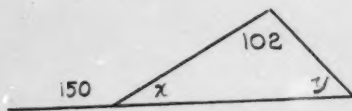
(a)



(b)



(c)

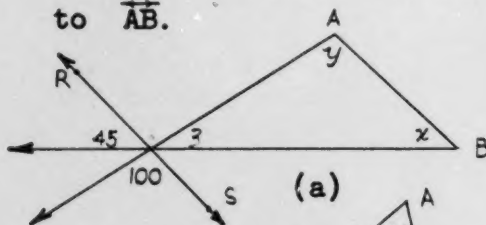


(d)

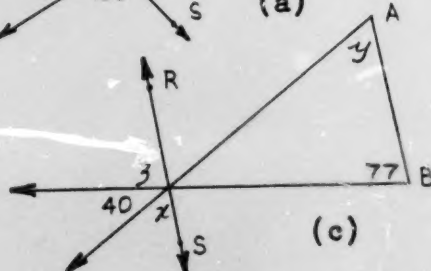


(e)

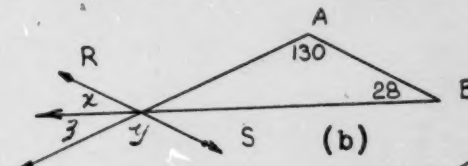
5. What is the measure of each angle of an equilateral triangle?
6. Suppose one angle of an isosceles triangle has a measure of 50.
- Find the measures of the other two angles.
 - Are two different sets of answers possible?
7. Find the measures of $\angle x$, $\angle y$, and $\angle z$. $\overleftrightarrow{RS}$ is parallel to $\overleftrightarrow{AB}$.



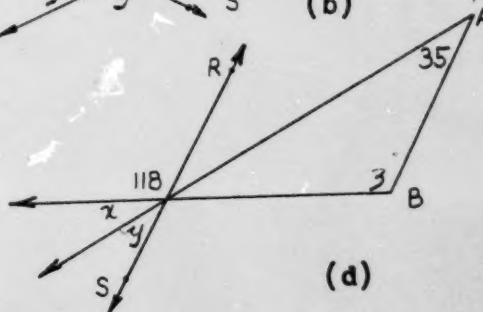
(a)



(c)

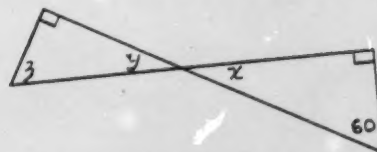


(b)



(d)

(a)



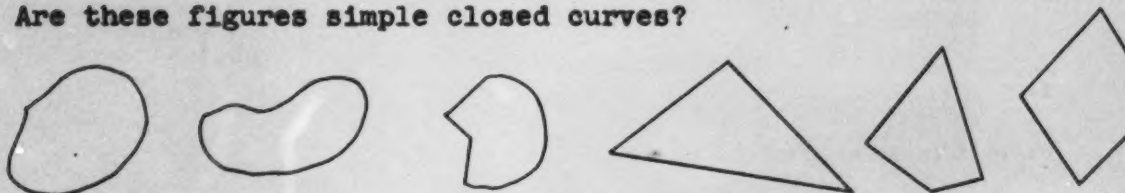
(b)

13-7. Parallelograms.

In Chapter 4 you discussed the following requirements for simple closed curves:

- a. The drawing starts and stops at the same point.
- b. No other point is touched twice by the pencil mark.

Are these figures simple closed curves?

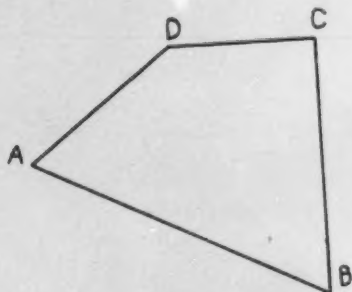


Notice these things about the last three figures:

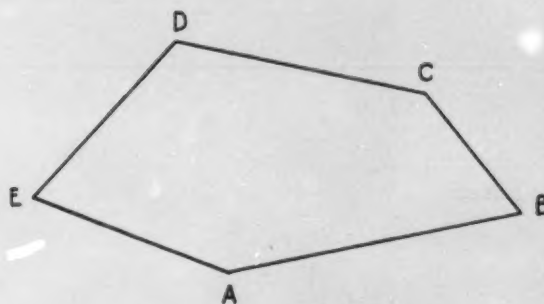
- a. They are simple closed curves.
- b. Each is the union of several segments.

A simple closed curve which is a union of segments is called a polygon. In later work you may sometimes see the word "polygon" used for curves that are not simple closed curves, but for us it will mean a simple closed curve.

Perhaps you have noticed already that the triangle is a certain kind of polygon. The name for a polygon depends on counting how many segments or sides form the figure. A polygon having four sides is called a quadrilateral, and a polygon with five sides is called a pentagon.



Quadrilateral

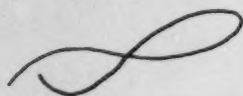


Pentagon

In a quadrilateral, two sides (segments) which do not intersect are called opposite sides. In the left-hand figure, $\overline{AD}$ is opposite $\overline{BC}$. $\overline{AB}$ and $\overline{DC}$ are opposite sides of quadrilateral $ABCD$.

Exercises 13-7a

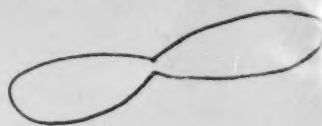
1. Which one of these three figures is a simple closed curve?



(a)



(b)

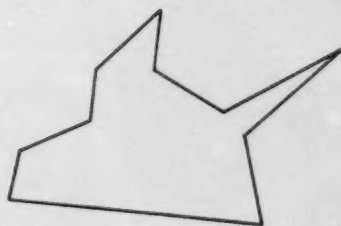


(c)

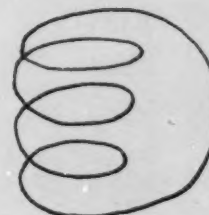
2. Which one of these three figures is not a simple closed curve?



(a)



(b)



(c)

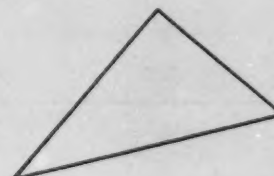
3. Which of these figures is a polygon? Which one is not? What requirement for polygon does it fail to meet?



(a)

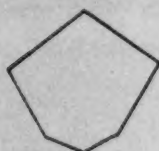


(b)



(c)

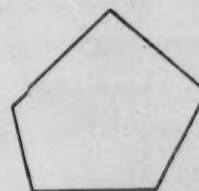
4. Which of these figures is a pentagon? Why?



(a)

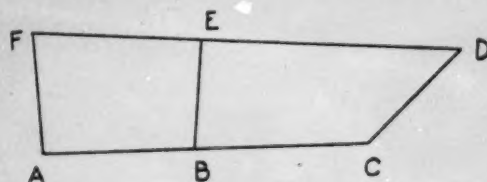


(b)

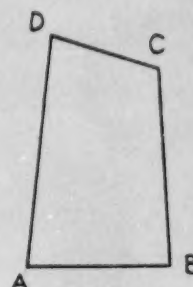


(c)

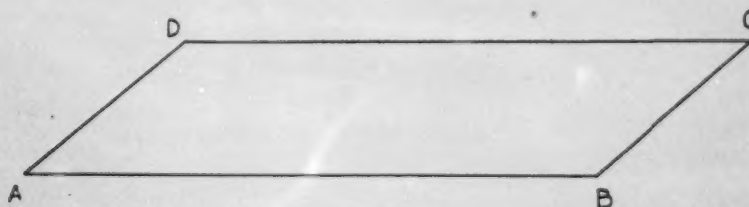
5. Name three quadrilaterals in this figure:



6. In the figure at the right:
- Name two sides in a pair of opposite sides.
 - Name a different pair of opposite sides.
 - Name two sides that do not intersect.
 - Name two other sides that do not intersect.



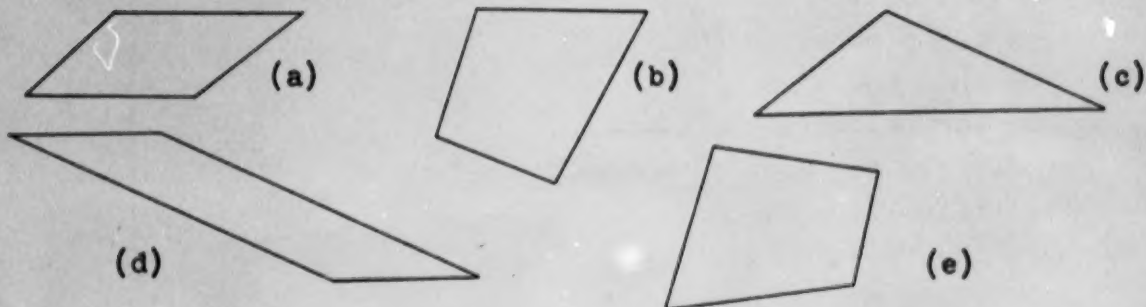
A very important kind of quadrilateral is the parallelogram. This is a quadrilateral whose opposite sides lie on parallel lines. The figure ABCD below represents a parallelogram. Name the sides in each pair of opposite sides.



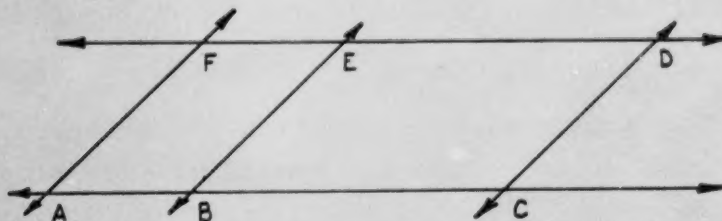
In the future, if two segments lie on parallel lines we shall speak of the segments as parallel. Thus we can say that the opposite sides of a parallelogram are parallel.

Exercises 13-7b

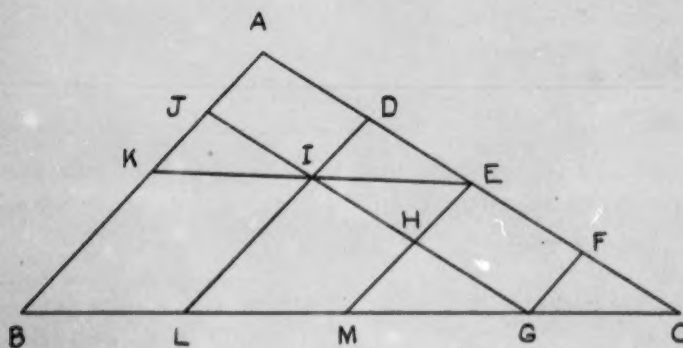
1. Which of the following figures are parallelograms, assuming that segments which appear to be parallel are parallel?



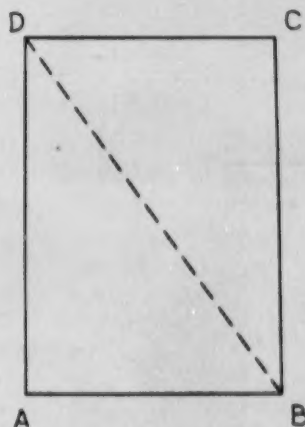
2. Name three parallelograms in this figure:



3. Name nine parallelograms in this figure:



4. The figure at the right represents a rectangular sheet of paper. (Notice that a rectangle is a special kind of parallelogram.) Fold the paper as shown and cut or tear it along the line of the fold. By laying one piece on the other, show that the area of triangle ABD and triangle BCD are equal. Is the area of one of the triangles equal to one-half the area of the parallelogram? Why?



Experiment Five

A Property of the Opposite Sides of a Parallelogram

What is needed: Ruler, pencil, paper.

What to do: 1. Find a pair of opposite sides in the parallelograms drawn below. Find the length of each side in the pair.

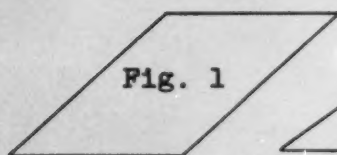


Fig. 1

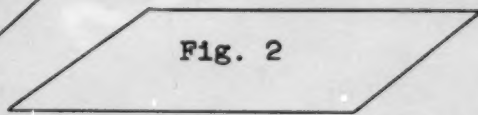


Fig. 2

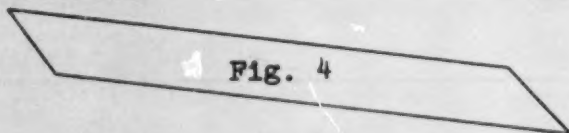


Fig. 4

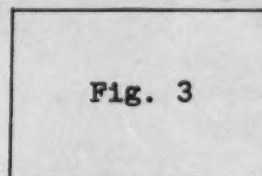


Fig. 3

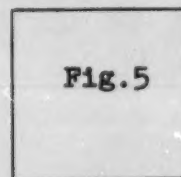


Fig. 5

2. Make a copy of the table below. Record the side measures in the table.

Number of the figure	Lengths of opposite sides:	
	First side	Second side
1	1 "	1 "
	$1\frac{1}{4}$ "	
2	2 "	

- Questions: 1. What have you discovered about the lengths of opposite sides of a parallelogram?
2. What requirement must a quadrilateral meet in order to be called a parallelogram?

This seems to be true:

Property 5. The opposite sides of a parallelogram are equal in length.

Exercises 13-7c

For these exercises you will need a quick way to draw parallelograms of different shapes and sizes. Here are two ways to do this. Follow the steps carefully with your compass and decide which of the two ways you like better. Be sure to answer the questions following the last step in each method.

Method I

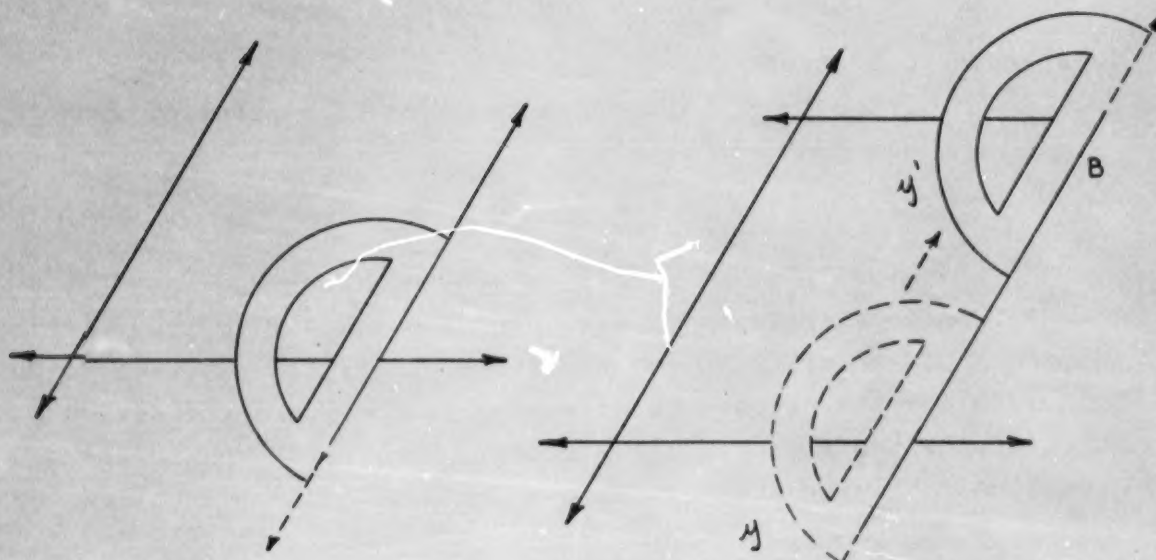
Step one. Draw two intersecting lines.



Step two. Measure angle x . A short distance to the right mark a point A . Slide the protractor to the right and make angle x' the same size as angle x .



Step three. Measure angle y . Mark a point B . Slide protractor so that you can make angle y' the same measure as angle y .



Questions:

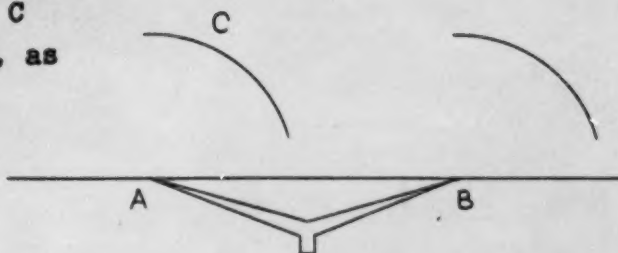
1. In step two, what kind of angle pair did you measure and mark? Are these angles equal in measure?
2. Do you have two parallel lines after this step? What property says so?
3. In step three, what kind of angle pair have equal measures? After this step, do you have another pair of parallel lines? How do you know?
4. Is the quadrilateral which results a parallelogram? Why?

Method II

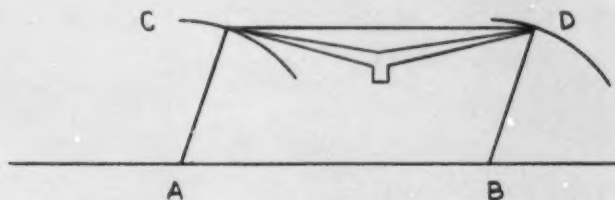
Step one. Draw a line and mark two points on it. Call these points A and B . Open your compass a little and make two curves as shown.



Step two. Open your compass the distance AB. Mark point C where you please, as shown.



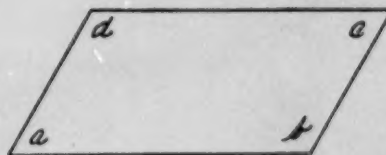
Step three. Placing compass point at C make a curve as shown. Draw the segments indicated.



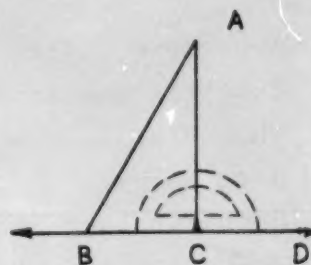
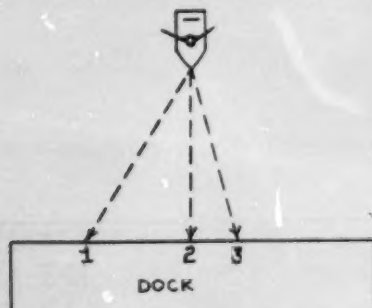
Questions:

1. Are the opposite sides of quadrilateral ABDC equal in length? How do you know?
2. What special kind of quadrilateral does figure ABDC seem to be?
3. Complete this statement: If the opposite sides of a quadrilateral are equal in length then the quadrilateral is a ? .
4. How many examples will you need to show that this statement is true?

1. Draw two parallelograms different in shape and size.
2. If your parallelograms are drawn carefully can you expect the opposite sides to be equal in measure? Why?
3. In the parallelogram at the right, $\angle a$, $\angle b$, $\angle c$, and $\angle d$ are called angles of the parallelogram. Are the angles actually part of the of the parallelogram or are they determined by the parallelogram? Why?
4. Label and measure the angles of the parallelograms you drew for Exercise 1. What curious fact can you discover about a certain pair of these angles? List the pairs that show this property and give their measures.



Suppose you are in a leaky rowboat not far from a long dock. You must get to the dock quickly before the boat sinks! Which of the paths marked in the picture should you take?



Do you think segment AC in the diagram at the right is the shortest distance from point A to line BC? Place your protractor as shown and measure angle ACB. Measure angle ACD.

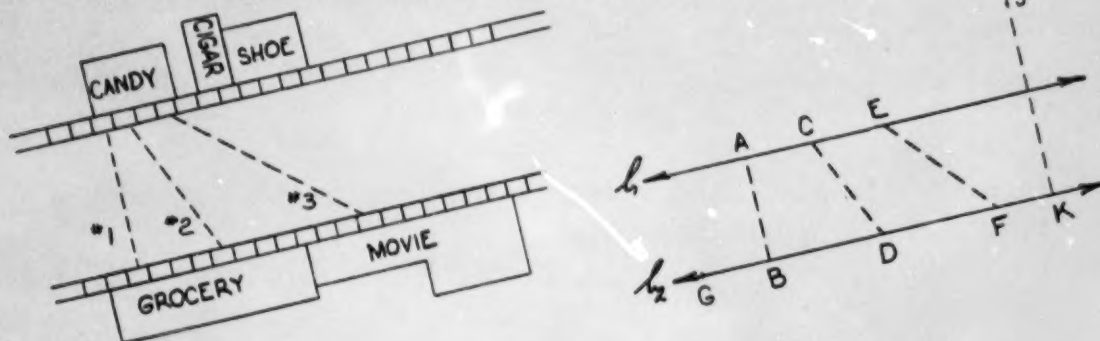
Do you agree that segment AC not only is the shortest distance from point A to the line BC but also forms two right angles with the line BC? In the future, the distance from a point to a line will mean the measure of a special segment. This

segment, like $\overline{AC}$, must meet the following requirements:

1. It must form right angles with the line.
2. One of its endpoints must be the given point.
3. Its other endpoint must be in the given line.

Which of the above requirements is not met by segment $\overline{AB}$?

Below is a picture of a street in town:



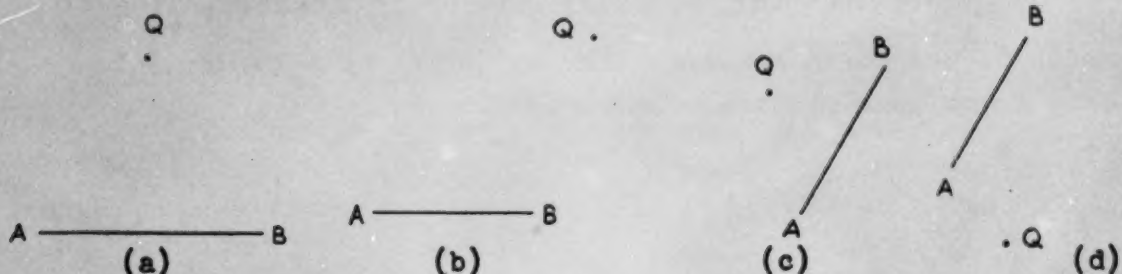
What is the number of the shortest path from the candy store to the grocery store? In the drawing at the right, which segment is the shortest distance between the parallel lines l_1 and l_2 ? What is the measure of angle $\angle ABG$? of angle $\angle BAC$? In the future, the distance between two parallel lines will be the measure of a special segment. This segment, like $\overline{AB}$, must meet the following requirements:

1. It must lie in a line that makes right angles with both of the parallel lines.
2. It must have an endpoint in each of the parallel lines.

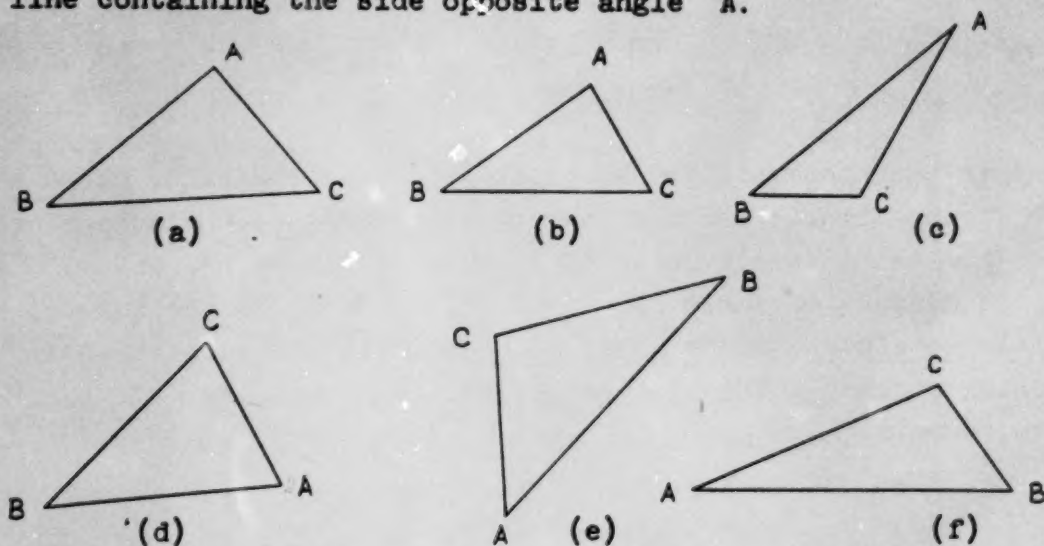
Which of the above requirements is not met by $\overline{EF}$? By $\overline{JK}$?

Exercises 13-7d

1. For each of these figures, find the distance between point Q and the line containing segment AB .

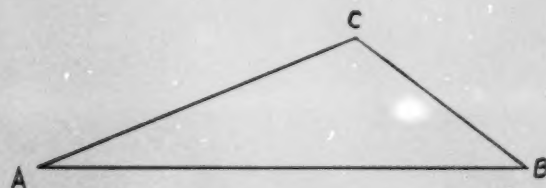


2. In each triangle, find the distance from vertex A to the line containing the side opposite angle A .



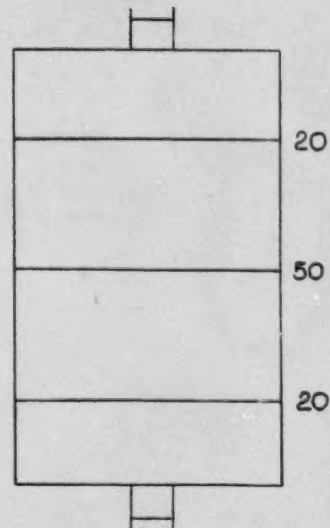
3. In the triangle below:

- How far is point A from the line containing $\overline{BC}$?
- How far is point B from the line containing $\overline{AC}$?
- How far is point C from the line containing $\overline{AB}$?



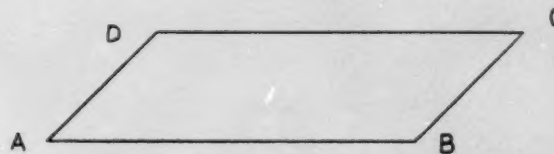
4. The yardlines in the football field at the right represent parallel lines. Draw segments which have measures as indicated:

- The distance between the goal lines.
- The distance between the twenty-yard lines.
- The shortest distance you can run in scoring a touch-down from the fifty-yard line.



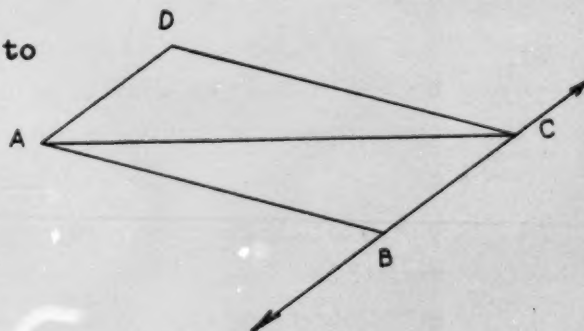
5. In the parallelogram at the right, find the distance between the sides:

- $\overline{AB}$ and $\overline{CD}$
- $\overline{AD}$ and $\overline{BC}$

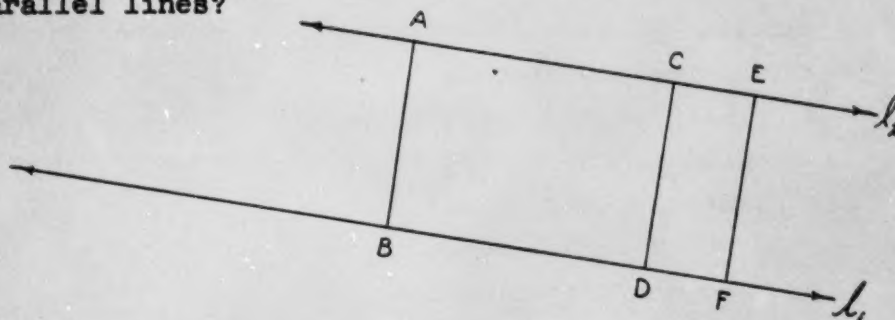


6. Make a copy of the figure at the right. Draw segments which have measures equal to the distance between:

- Point A and $\overline{BC}$
- Point B and $\overline{AD}$
- $\overline{AB}$ and $\overline{CD}$
- $\overline{AD}$ and $\overline{BC}$
- Point C and $\overline{AD}$



In the drawing of two parallel lines below, several segments are labeled. Is the measure of one of these segments the distance between the parallel lines?



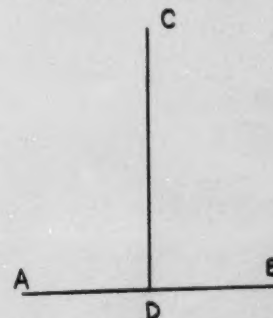
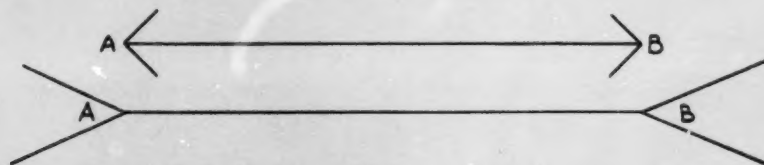
Measure segments AB , CD , and EF . Do these segments have the same measure?

You may have decided that these segments have a common measure. If so, then you can say that the distance between the parallel lines is the same for each segment. Suppose many more segments are drawn with their endpoints in l_1 and l_2 so that each segment lies in a line forming right angles with l_1 and l_2 . Will these segments have the measure common to $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$? If so, you can say, "It seems to be true that the distance between two parallel lines does not change."

We can't be sure that the last statement is true, but it does seem to be true. We can't be sure because the reasons for believing that the statement is true are not good reasons. The reasons are:

1. The distances look the same.
2. We tested the idea with many correctly drawn segments and found the distance unchanged for each case.

Consider reason number one. To show that this is a poor reason for belief, look at the figures below:



Without measuring, name the longer segment in each figure. Now measure the segments. Are you as certain now about the lengths as you were before? Can you be sure about the lengths of segments by simply looking at them?

Now consider reason number two. If testing many times is a way to make sure of a statement, let's test this number statement many times:

If you:

1. Multiply a counting number by the next smaller counting number,
2. And add 41 to the product,

Then the final result will always be a prime number.

The same idea may be stated this way: For a counting number n the expression $n(n - 1) + 41$ always represents a prime number. Do you think the statement true? How many tests are needed to show the statement true?

Here are forty tests of the above number statement:

After checking the table on the right are you sure that the number statement is true? It is very likely that you have tested the number statement many more times than you did our geometric statement.

Now try just one more test. Try the counting number 41. Is the result a prime? Try dividing the result by 41. Now answer the question at the beginning of this paragraph!

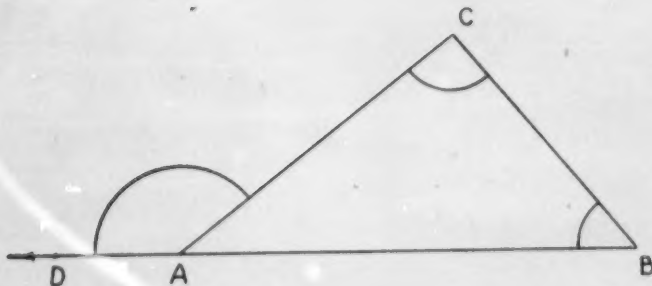
Number	Expression	Result
1	$1(0) + 41$	41
2	$2(1) + 41$	43
3	$3(2) + 41$	47
4	$4(3) + 41$	53
5	$5(4) + 41$	61
Complete this table through the number 39		
40	$40(39) + 41$	1601
41	$41(40) + 41$	?

To make sure of a certain property, mathematicians follow a kind of plan. This plan is called a proof. All of the properties you have studied in your experiments will be proved in future work in geometry. While you will not be expected to do any proofs of your own this year, in time you will become quite good at making proofs.

One of the main parts of a proof lists certain rules or agreements that must be made before continuing the discussion. As seen in the sample proof below, the name of this part is "Facts Already Agreed Upon".

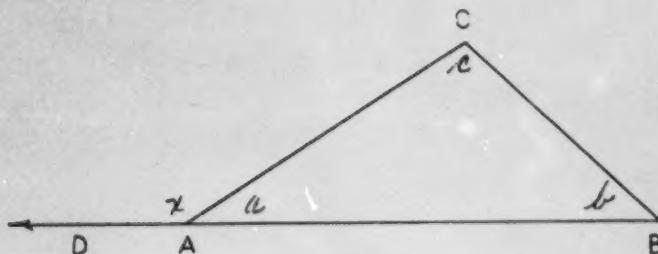
Why must there be such a section? Suppose your hockey team has just scored a goal. "The score doesn't count!", shouts a player on the other team. "I just made up a rule that all scores must be made while running backwards!" Can you give some reasons why certain agreements should be made before starting a game? For many of the same reasons there must be a list of "Facts Already Agreed Upon".

The following proof is planned to show that the measure of angle CAD is equal to the measure of angle A added to the measure of angle C. The following drawing represents the angles in question:



A Proof

The Measure of $\angle CAD$ Is Equal to
the Measure of $\angle B$ Added To the Measure of $\angle C$



Facts Already Agreed Upon:

1. The requirements for supplementary angles.
2. Property 4. The sum of the measures of the angles in a triangle is 180.
3. The letters a , b , c , represent the measures of the angles of the triangle. The letter x represents the measure of $\angle CAD$.
4. $180 - a$ is that number which added to a gives 180.

Statements	Why the Statement is True
1. $x + a = 180$	1. Angle CAD and angle CAB meet the requirements for supplementary angles. Item 3 of the "agreed upon" section.
2. $180 - a = x$	2. Item 4 in the "agreed upon" section.
3. $(b + c) + a = 180$	3. Property 4. The sum of the measures of the angles in a triangle is 180.
4. $180 - a = (b + c)$	4. Item 4 in the "agreed upon" section.
5. $x = b + c$	5. Steps 2 and 4 show that x and $(b + c)$ are two names for the same number, $180 - a$.

13-8. Areas of Parallelograms and Triangles.

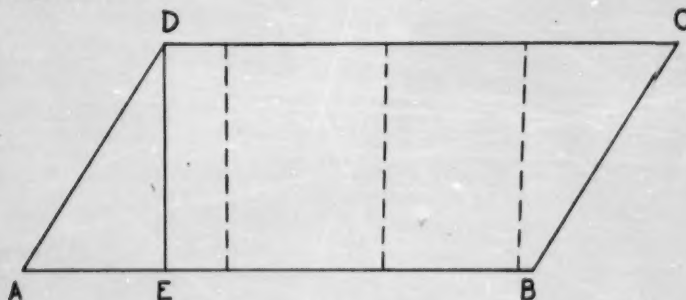
A parallelogram as you have learned, is a quadrilateral (a four sided figure). The opposite sides lie on parallel lines. A rectangle is a quadrilateral. Is a rectangle a parallelogram?

As you have already discovered, the opposite sides of a parallelogram have the same length. Which pairs of sides in the figure below have the same lengths?

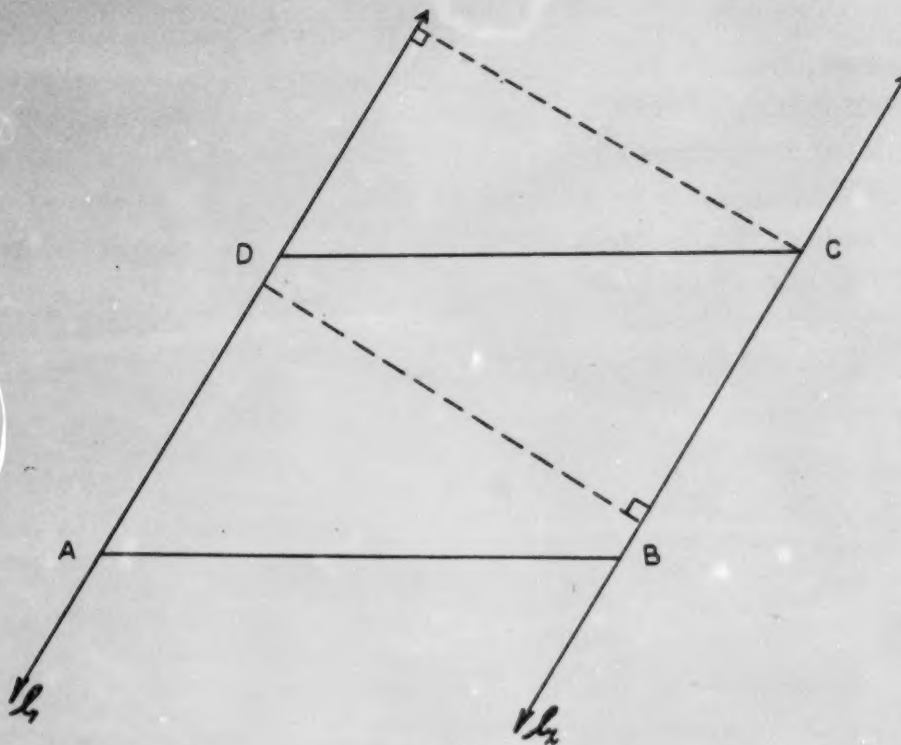


A polygon is the union of segments. The endpoints of the segments which have the polygon as their union are called the vertices of the polygon. Points A, B, C, and D, are the vertices of the parallelogram above.

In the figure below, segment DE is called an altitude of the parallelogram.

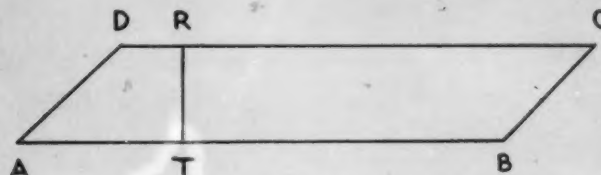


A parallelogram has many altitudes. The dotted segments represent altitudes of the same parallelogram but they are in different positions. When we wish to speak of the measure of an altitude, we refer to the number as the altitude. You may have noticed that DE is also called the distance between the parallel lines AB and DC. The altitude of a parallelogram is the distance between a pair of parallel lines of the parallelogram.



We see then, that a pair of opposite sides of a parallelogram determines a set of altitudes and that the other pair also determines such a set. In each set the altitudes have the same measure. The measure of an altitude in one set is not always the same as the measure of the altitude in the other set.

The sides between which the altitudes are drawn are called bases of the parallelogram. In the figure below, we say that altitude $\overline{RT}$ is drawn between the bases $\overline{AB}$ and $\overline{CD}$. Sometimes we say that $\overline{AB}$ is the base to which the altitude is drawn or that $\overline{CD}$ is the base to which the altitude is drawn.



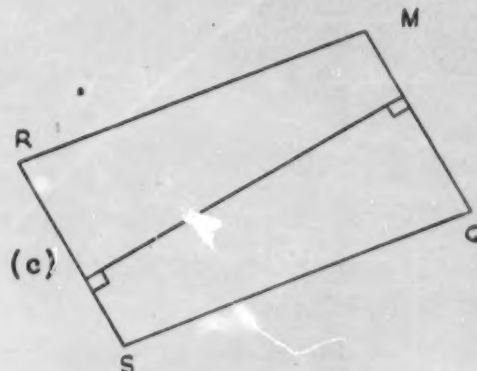
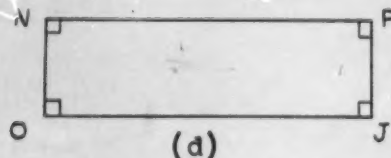
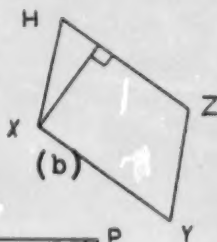
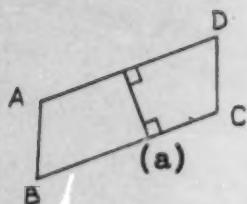
Exercises 13-8a

1. Polygon ABCD is a parallelogram.

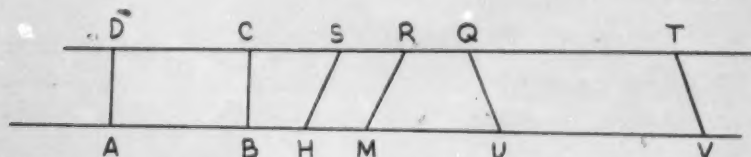
- List the vertices.
- List the sides.
- Name a pair of opposite sides.
- Is the distance between opposite sides the same whether measured between $\overline{AB}$ and $\overline{CD}$ or between $\overline{AD}$ and $\overline{BC}$?



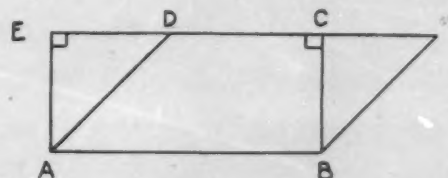
2. Name one base for each altitude in the following parallelograms.



- 3.



- Name three parallelograms in the figure above.
 - Is the measure of an altitude in one of these the same as the measure of an altitude in one of the others? Why?
4. In the figure at the right:
- Name two parallelograms.
 - Name two segments either of which is an altitude for both parallelograms.

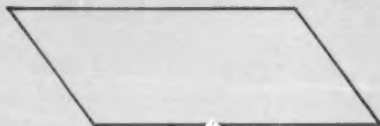


Experiment Six

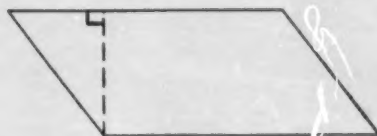
Finding the Area of a Parallelogram.

What is needed: Scissors, pencil, paper, straight-edge.

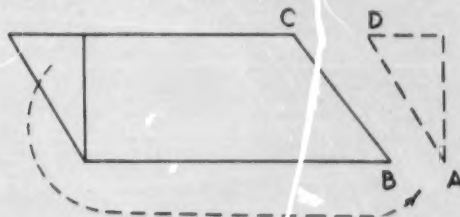
What to do: 1. Draw a parallelogram. Cut it out.



2. Draw an altitude as shown. Cut on the dotted segment.



3. Without lifting either piece from your desk, slide the triangular region to the right and place it in the position shown.



4. Slide the two pieces together so that point D falls on point C and point A falls on point B.

Questions:

1. What special polygon is suggested by the two pieces in their final positions?
2. Is the area of this polygon the same as the area of the parallelogram with which you began this experiment? Why?

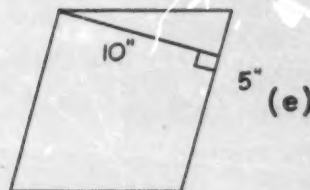
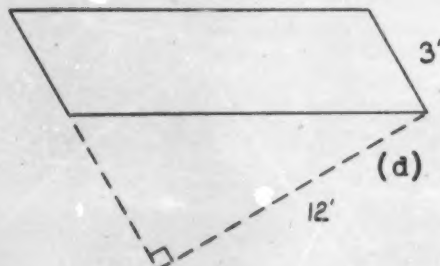
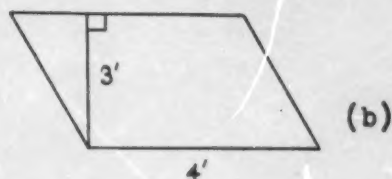
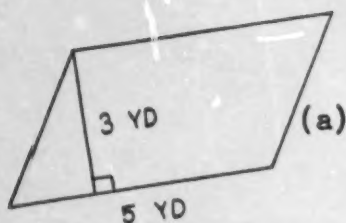
3. Is the measure in the width of the rectangle the same as the measure of an altitude in the parallelogram?
4. Is the measure in the length of the rectangle the same as the measure of the base of the parallelogram to which the altitude is drawn?
5. In order to find the area of a parallelogram will it always be necessary to cut it up as we have done? Why?

What seems to be true:

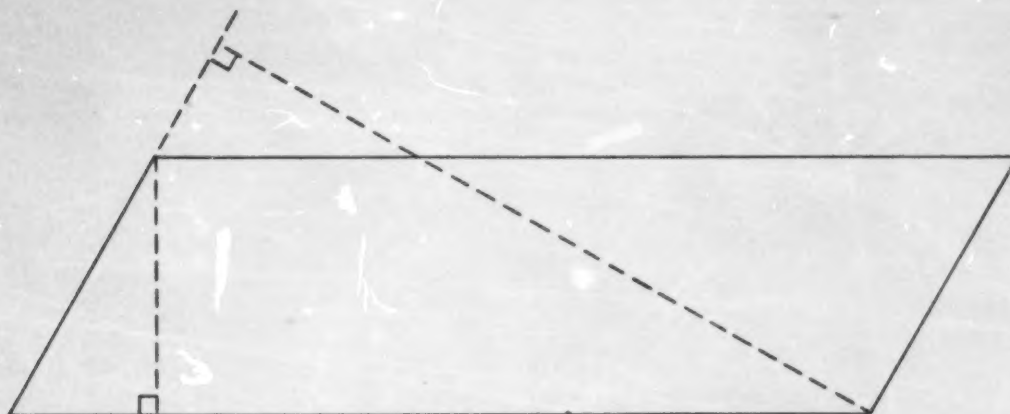
Property 6. The area of a parallelogram is the product of the measure of ? and the measure of the ? to which it is drawn.

Exercises 13-8b

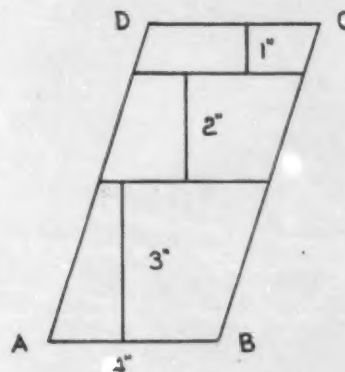
1. Find the areas of the following parallelograms:



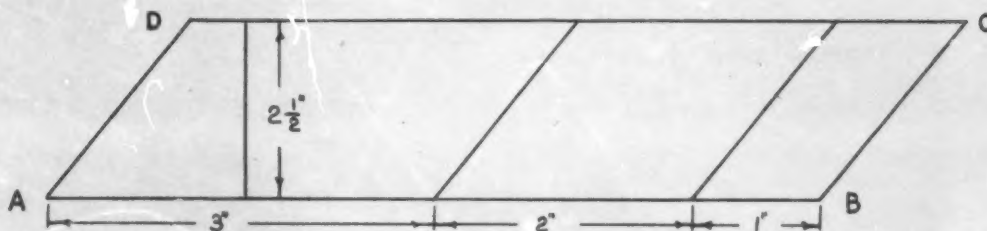
2. Measure the bases and altitudes of the parallelogram below and use them to find the area in two ways. How well do your results agree? Since measurement is approximate, they may not be exactly the same, but should be close.



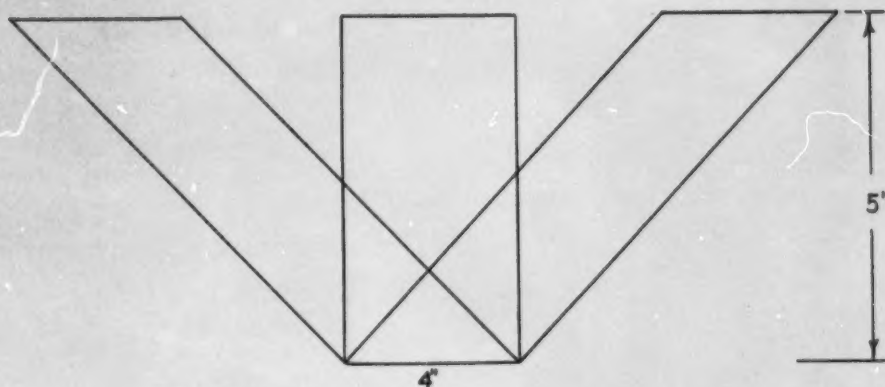
3. Find the area of parallelogram ABCD in two ways by using Property 6 (a) three times (b) once.



4. Find the area of parallelogram ABCD in two ways.
 a. By using Property 6 three times.
 b. By using Property 6 once.



5. Show that these parallelograms have the same area:

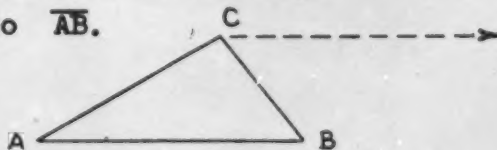


- *6. Look at the parallelogram and rectangle below which have bases of equal length and altitudes of equal length.

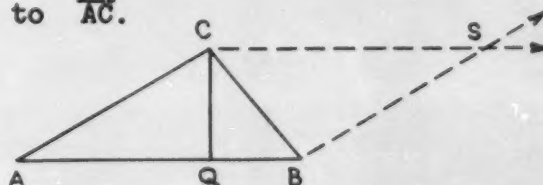


Cut the closed region of the parallelogram into pieces that can be reassembled to form the closed rectangular region.
(First make a copy of the parallelogram.)

Consider any triangle ABC as shown at the right.
Through C draw a line parallel to $\overline{AB}$.



Through B draw a line parallel to $\overline{AC}$.

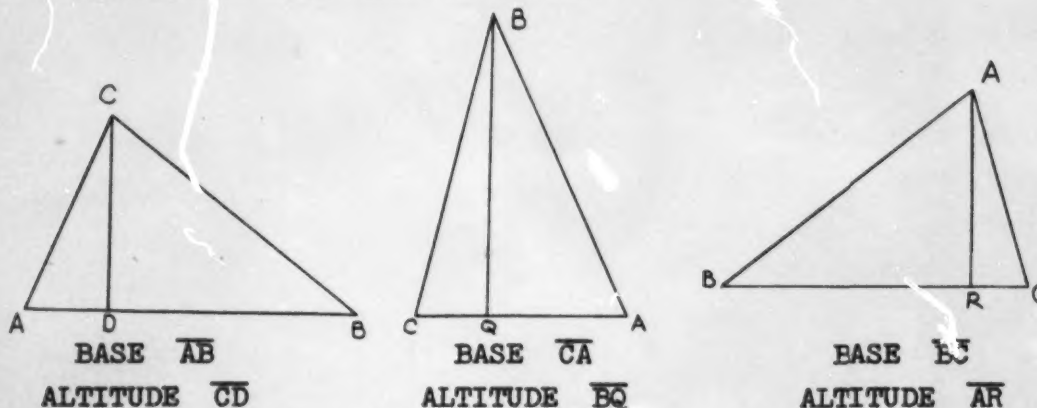


Label the point of intersection of the lines you have drawn with the letter S.

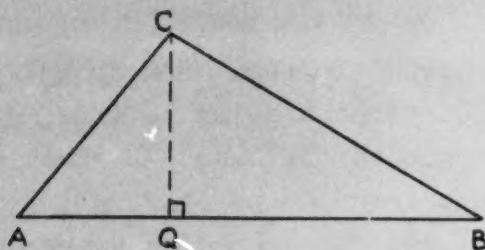
The figure $ABSC$ is therefore a parallelogram. How do we know this? The segment CQ through C perpendicular to line $\overline{AB}$ is called the altitude of the triangle to the base $\overline{AB}$. The measure of altitude $\overline{CQ}$ is the distance from C to line $\overline{AB}$.

In Section 7 (Exercise 4 of 13-7b) you discovered that the triangular regions ABC and SCB are the same size. Since the area of the parallelogram is completely made up of the areas of these two triangular regions, it follows that the area of triangle ABC is one-half the area of the parallelogram $ABSC$. Copy and complete the following statement: "The number of square units in the area of a triangle is ? the ? of the number of linear units in the ? and the number of linear units in the ? to this base."

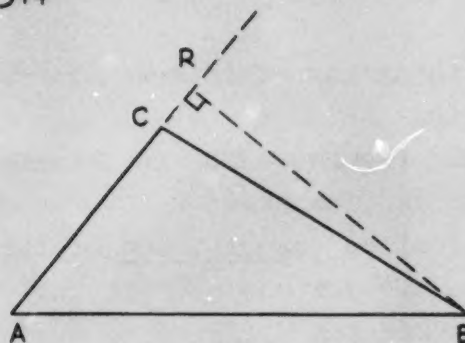
Any side of a triangle may be considered a base. The following figures show this to be so:



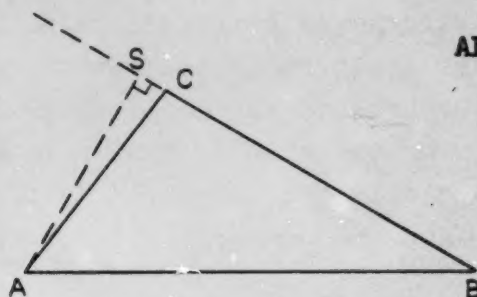
In the three figures above the base was shown as horizontal in each case. The base does not always lie in a horizontal position. The following figures show this:



BASE $\overline{AB}$
ALTITUDE $\overline{CQ}$



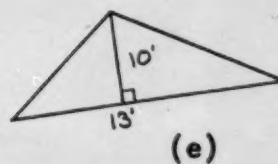
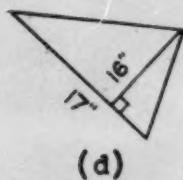
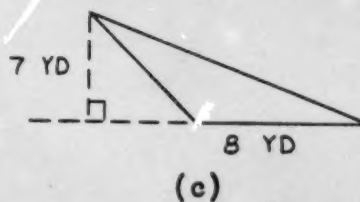
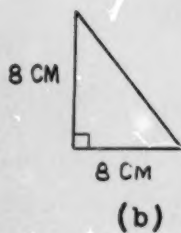
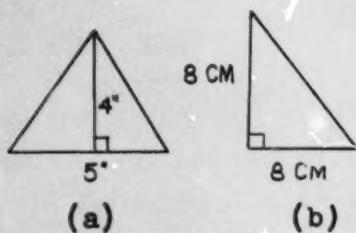
BASE $\overline{AC}$
ALTITUDE $\overline{BR}$



BASE $\overline{BC}$
ALTITUDE $\overline{AS}$

Exercises 13-8c

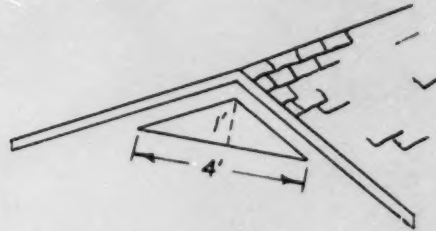
1. Find the areas of the triangles shown, using the measures given.



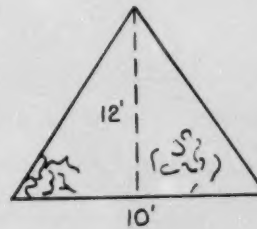
2. A right triangle has sides of 5 yards and 12 yards as shown. Find the number of square yards in its area.



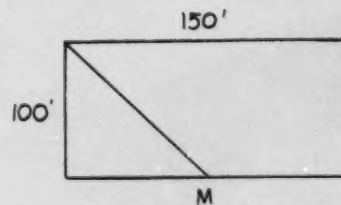
3. A certain house attic must have openings near the roof which have a total area of 12 square feet. This is needed for good cooling of the attic. The house has two openings like that in the drawing. Is this attic well ventilated?



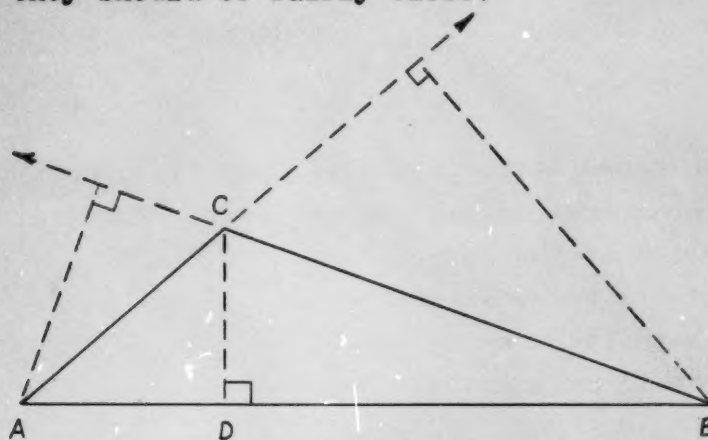
4. A stained-glass window in a church is in the shape of a triangle. If its measures are as shown, what is the total cost of the window if the cost per square foot is \$50?



5. A man owned a rectangular lot 150 ft. by 100 ft. From one corner A, there was a path to the point M in the center of the longer opposite side. Find the area of the two pieces of the lot.

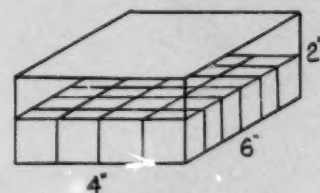
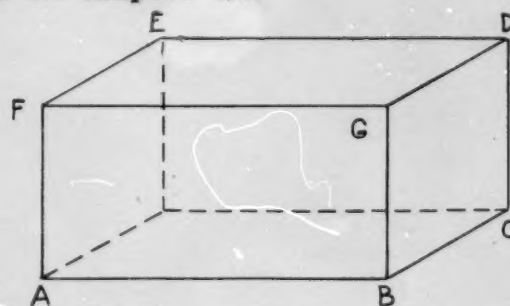
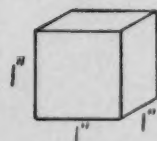


6. Measure the bases and altitudes in the triangle ABC and thus find the area of this triangle in three different ways. How well do your results agree? They may not be exactly the same but they should be fairly close.



13-9. Right Prisms.

In Chapter 12 you studied rectangular prisms. The following figures are also in Chapter 12.



In this chapter you will study other kinds of prisms.

Imagine holding in one hand a model of a table with a triangular top and holding in the other hand a triangular piece of cardboard as in Figure 1. The cardboard and the table top are exactly the same size and shape.

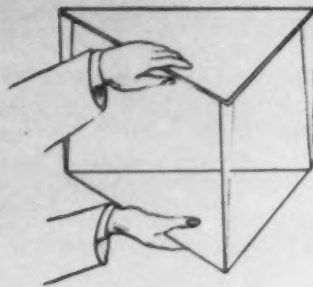


figure 1

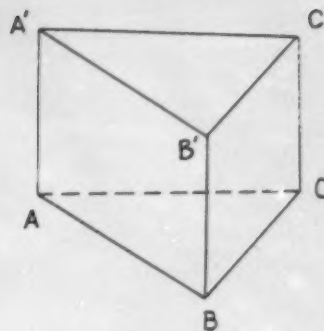


figure 2

Figure two represents segments which are suggested by the table legs, the edges of the top, and the edges of the cardboard. What are the names of the segments suggested by the edges of the table top? What are the names of the segments suggested by the table legs? By the edges of the cardboard?

Not only segments but several planes are suggested by the "table and cardboard" picture. The table top suggests a plane that is parallel to the plane of the cardboard. If the plane determined by points A' , B' , and C' is parallel to the plane determined by A , B , and C then the labeled geometric figure needs to meet just one more requirement in order to be called a triangular right prism. The requirement is that the quadrilaterals determined by $ABB'A'$, $BCC'B'$, and $ACC'A'$ must be rectangles.

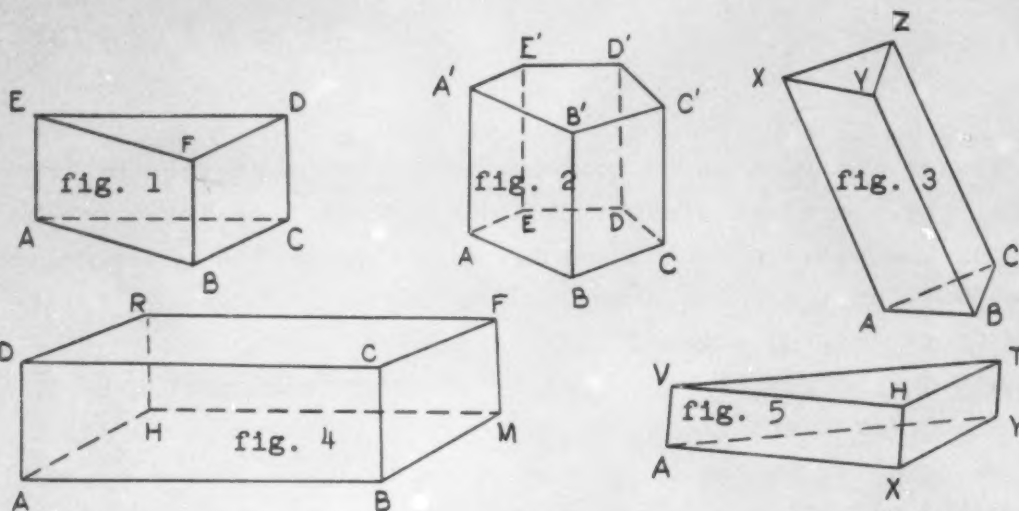
The closed rectangular regions which are determined by:

1. the edges of the table top
2. the edges of the cardboard and
3. the edges which the legs suggest

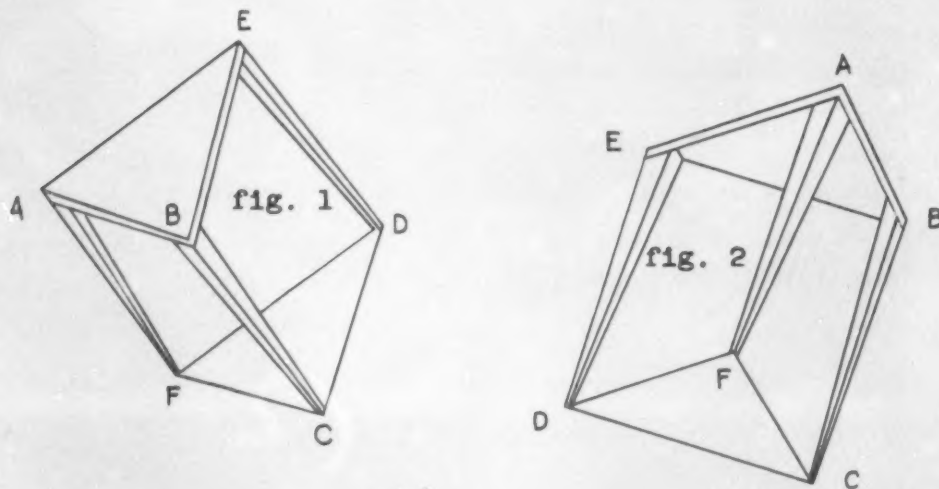
are called faces. The triangular regions are faces too, but they are called bases when they are the particular faces being discussed. Points A , B , C , A' , B' , and C' are called the vertices of the triangular right prism.

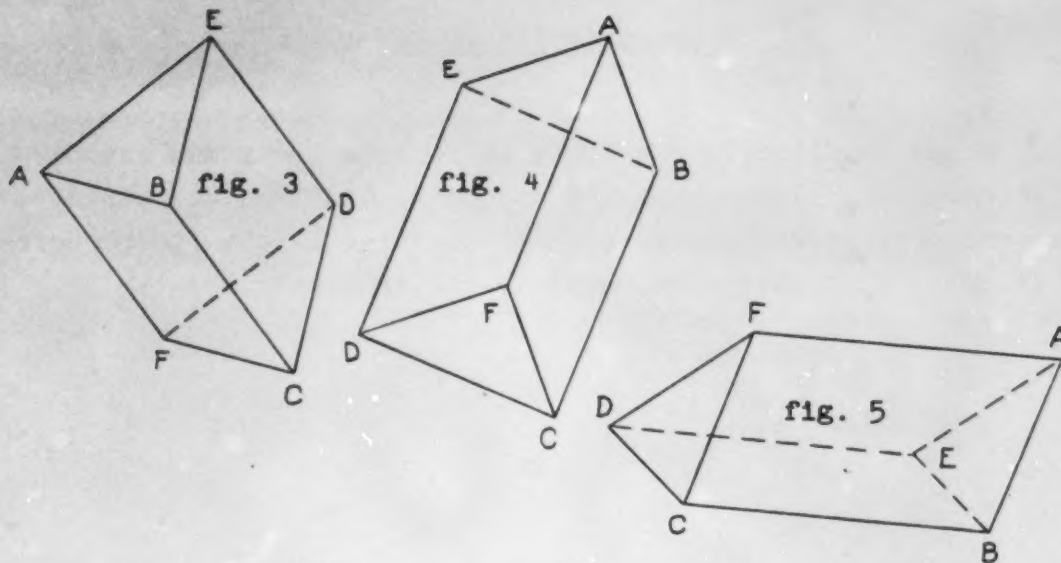
Exercises 13-9a

1. For each of these right prisms do the following:
- Write the name in words.
 - List the vertices.
 - List the rectangular faces.
 - Name both bases, with letters of their vertices.



It may seem from the drawing of the table that the bases of a right prism must lie in horizontal planes. The drawings below are intended to show that the bases of a right prism may lie in planes that are not horizontal.

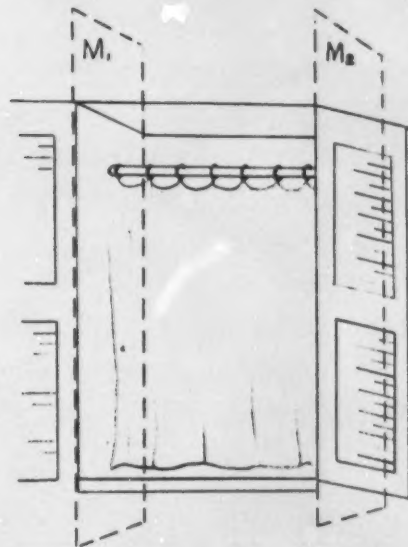




In the drawing at the right, the sides of the window frame are parallel planes marked M_1 and M_2 . The curtain rod can be described as fitting exactly between the parallel planes.

The measure of the rod represents the distance between the two parallel planes M_1 and M_2 .

The rod will fit exactly between the planes M_1 and M_2 in many other positions than that shown. This suggests that the distance between parallel planes, like the distance between parallel lines, is always the same number.

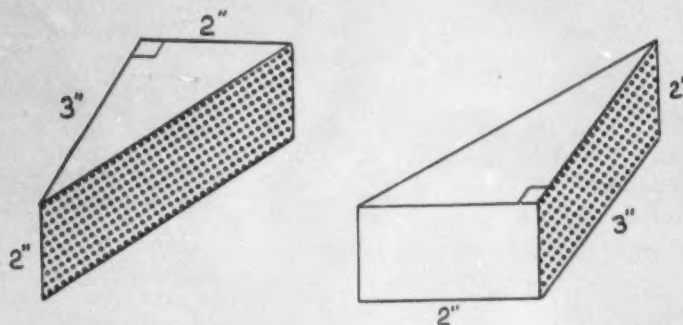


In a prism the distance between the parallel planes which contain the bases is called the height of the prism. In a right prism the measure of an edge suggested by one of the table legs may be used as the height of the prism.

Class Exercise 13-9b

Volume of a Right Prism

One way to discover new ideas is to take a certain example for which measures are given and use these measures to obtain at least one test of the new idea. For example, in the figure below are two pieces of cake with the measures as given:



Notice that these pieces are triangular prisms of the same shape and size. By placing them together in the following way another kind of prism is formed:

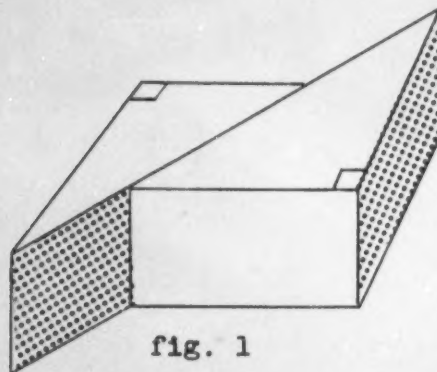


fig. 1

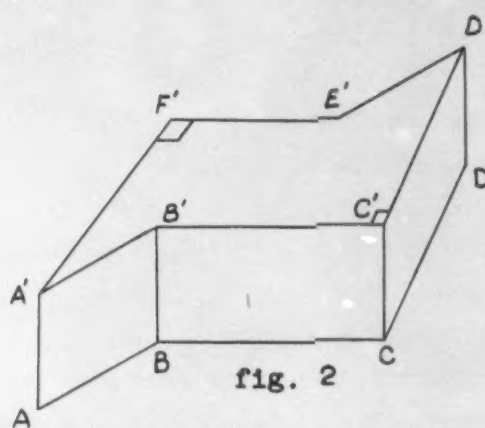
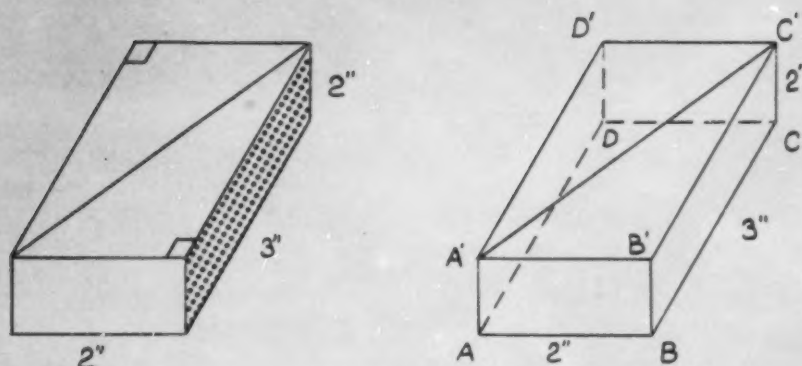


fig. 2

What reasons are there for thinking figure two is a prism? It is not likely that you know the name of the prism above but if we slide the pieces of cake along a little further we get a prism that you should be able to name.



Find the volume of the rectangular prism. Is the volume 12 cubic inches? Since two of the triangular prisms make up the rectangular prism, what is the volume of one triangular prism?

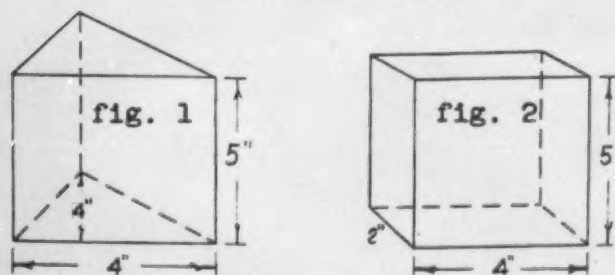
There is another way of finding the volume of one of the triangular prisms without using a rectangular prism at all. Try multiplying the height of the triangular prism by the area of its base. Do you get the same result as before?

While you have tested the following statement only once, you will find in future study of geometry that the statement is true for all prisms.

The number of cubic units of volume in the prism is the product of the number of square units of area in the base and the number of linear units in the height.

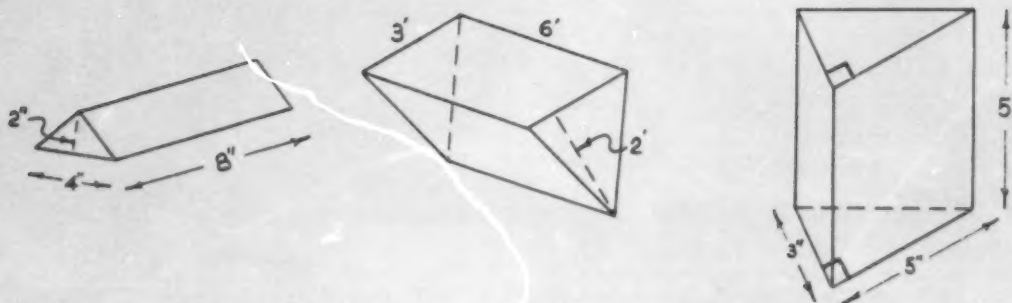
Exercises 13-9c

1.

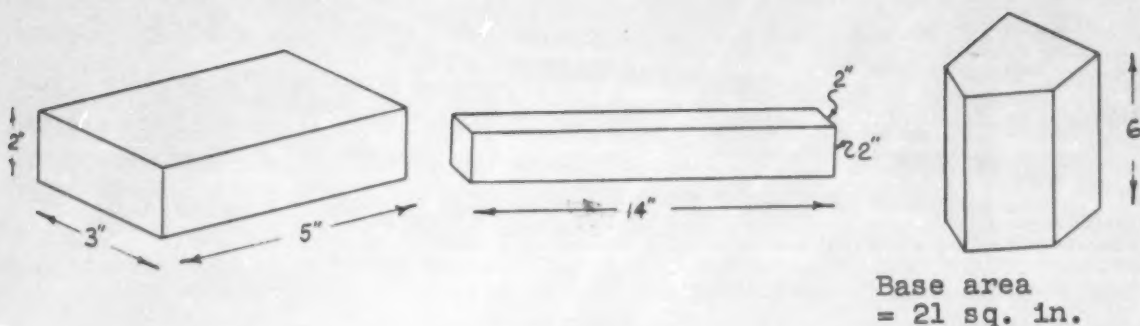


- Find the area of the base in Figure 1.
- Find the area of the base in Figure 2.
- What is the height of each right prism?
- What is the volume for each figure?

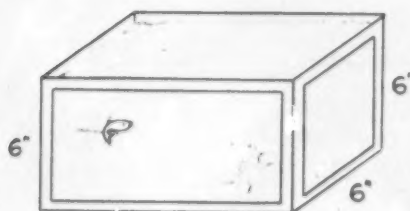
2. Find the number of cubic units of volume for each of the right prisms shown below:



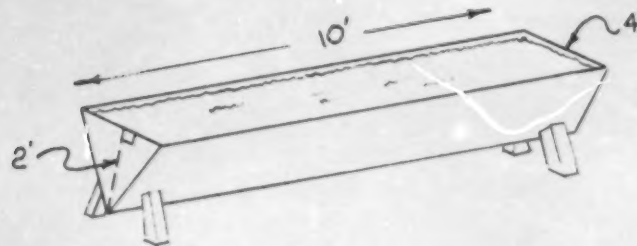
3. Find the volume for each right prism below:



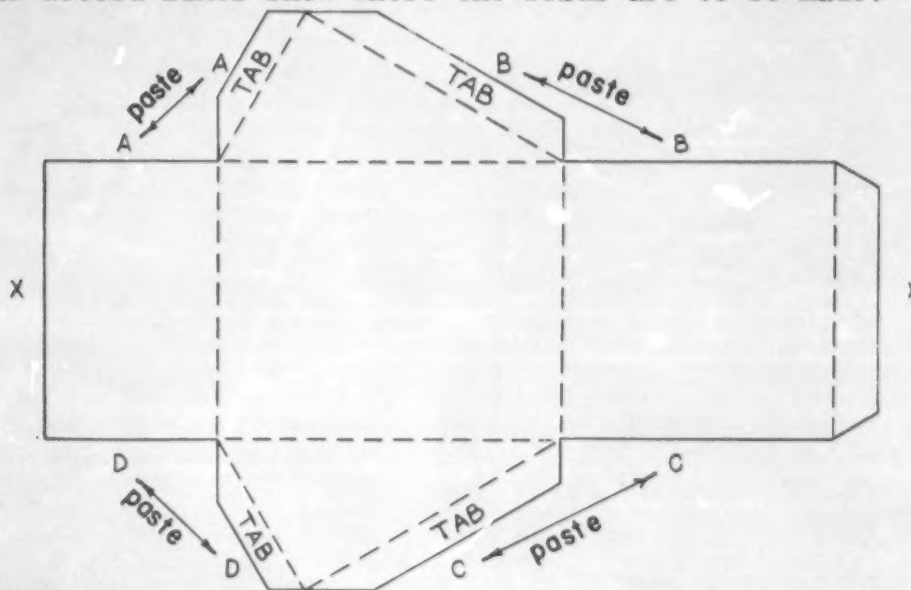
4. A fish tank must have at least 12 cubic inches of water for each fish of a certain kind.
- What is the volume of the fish tank?
 - What is the greatest number of fish allowed in the tank?



5. At the zoo some animals in a certain cage drink a total of at least fifty cubic feet of water a day. Can the tank below hold a day's supply of water for these animals?

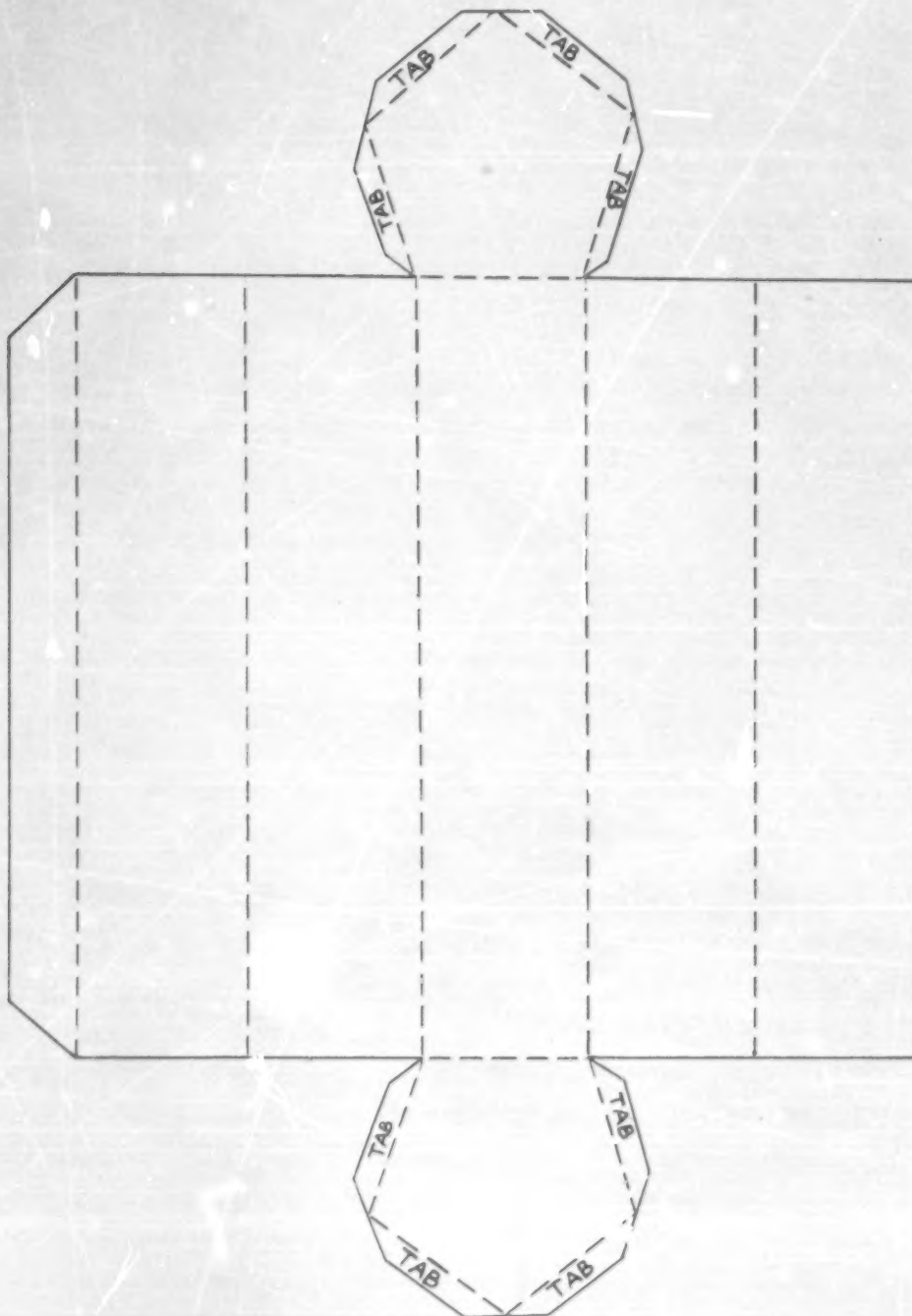


6. Make a copy of the following pattern. The copy must be drawn on stiff paper. The letters show which parts are to be pasted. The dotted lines show where the folds are to be made.



- What geometric figure is represented by this model?
- To find its volume a certain distance is needed. Draw on the base a segment whose measure is this distance.
- What is the measure of this segment?
- Find the volume of this right prism.

7. Fold and paste a copy of this pattern. What kind of geometric figure is represented by this model?
- a. If the area of the base is 6.25 square inches, what is the volume of this figure?



13-10. Summary.

In Chapter 13, a few simple ideas about two intersecting lines have led to a great number of geometric ideas about polygons and prisms. At the start you studied angles for the most part--how they were determined, what to name certain pairs of them, and how their measures depended on the positions of the lines by which they were determined. These ideas about angles were used to discover facts about special polygons and then these polygons were studied together in a special figure called a right prism.

Important angle pairs studied in the first part of this chapter are:

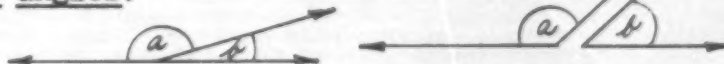
1. Adjacent angles.



2. Vertical angles.



3. Supplementary angles.

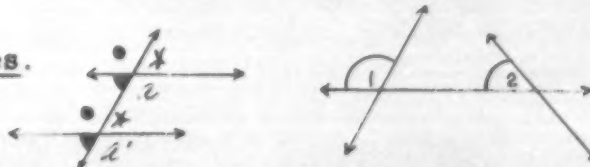


You discovered this property of vertical angles:

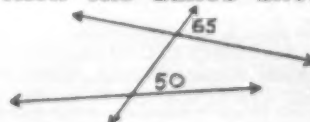
Property 1. If two lines intersect, the two angles in each pair of vertical angles formed have equal measures.

This property helped in working exercises about three lines intersecting in a plane. In talking about these lines you found it helpful to know about:

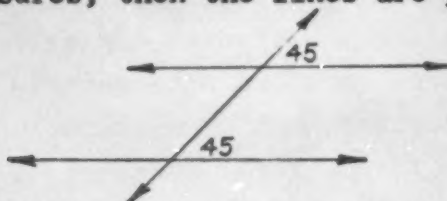
1. Corresponding angles.



2. Property 2. If a transversal intersects two lines in the same plane and a pair of corresponding angles have different measures then the lines intersect.



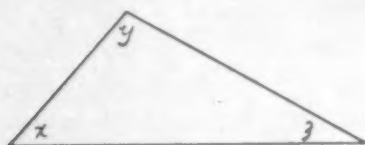
3. Property 2a. If a transversal intersects two lines in the same plane and a pair of corresponding angles have equal measures, then the lines are parallel.



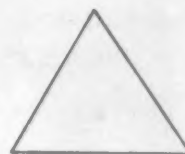
The properties just mentioned are true for certain special triangles. The property met in Section 4 is true for all possible triangles.

Property 4. The sum of the measures, in degrees, of the angles of any triangle is 180.

In the experiment which resulted in this property, you saw that a letter may be used over again in the same diagram if a small dash mark was placed over it. (x') By doing this, it was easier to find matching parts in figures that were alike in size and shape.



Section 7 spoke of special simple closed curves called polygons as the union of several segments.

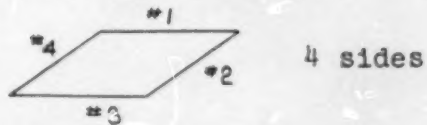


A special quadrilateral studied is the parallelogram.



Some important facts to remember about a parallelogram are:

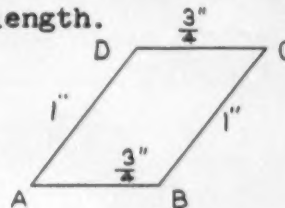
1. It is a quadrilateral.



2. The opposite sides are parallel.



3. Property 5. The sides in a pair of opposite sides of a parallelogram are equal in length.



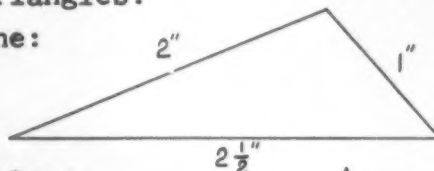
You learned of a new way to think up ideas with which to experiment. The plan is to interchange the "if clause" and the "then clause" of a statement to get a new idea or statement called a converse.

<u>Statement</u>	<u>Converse</u>
If $\angle a$ and $\angle b$ are vertical angles	If $\angle a$ and $\angle b$ have equal measures
then $\angle a$ and $\angle b$ have equal measures.	then $\angle a$ and $\angle b$ are vertical angles.

The idea of forming converses was helpful in Section 5 where you discovered properties about some special triangles.

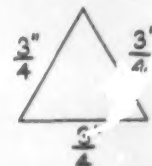
Special triangles:

1. Scalene:



2. Isosceles:

- a. Equilateral:



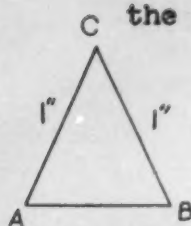
In this section, a statement and its converse took the form of two properties.

Statement:

Property 3. If two sides of a triangle have the same length then the angles opposite these sides have equal measure.

Converse:

Property 3a. If two angles of a triangle have equal measures, then the sides opposite these angles have the same length.

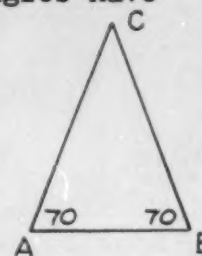


If $m \overline{AC} = m \overline{BC}$
Then $m \angle A = m \angle B$

Property 3

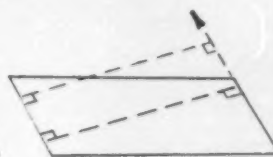
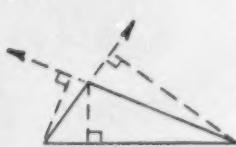
If $m \angle A = m \angle B$
Then $m \angle \overline{AC} = m \angle \overline{BC}$

Property 3a



You also learned in this section that the shortest segment from a point to a line is perpendicular to the line and that two parallel lines have an unchanging distance apart. Then you found that, while your experiments discovered important ideas and tested them, a certain plan called a proof is used in making sure a particular idea is true for all possible cases.

Section 8 showed how the distance between a point and a line or the distance between two parallel lines is the measure of a segment called an altitude of a triangle or of a parallelogram.



1. In a triangle there are three altitudes different in measure.
2. In a parallelogram there are many altitudes but only two possible measures for them according to the pair of opposite sides between which they are drawn.

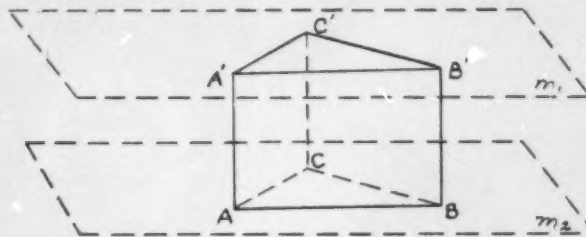
3. Any side of a triangle or parallelogram may be used as the base for an altitude.

You were able to find the area of a parallelogram and the area of a triangle.

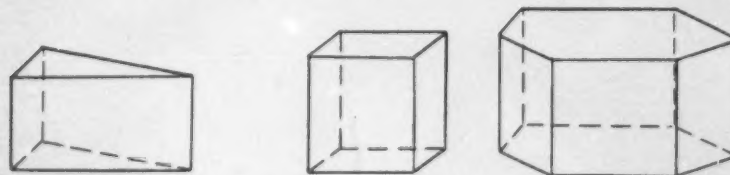
- a. The area of a parallelogram is the product of the measure of an altitude and the measure of the base containing an endpoint of the altitude.
- b. The area of a triangle is the product of the measure of an altitude and the measure of the base containing an endpoint of the altitude.

Finally in Section 9, all of the ideas in previous sections were used to study prisms. Some important facts to remember about a right prism are:

1. Its bases must lie in parallel planes.



2. Its bases must be exactly alike in size and shape.
3. All faces, excepting the bases, must be rectangles. The bases in a rectangular prism are also rectangles.
4. The name of a particular right prism depends on the number of sides of a base.



5. The number of cubic units of volume in a prism is the product of the number of square units of area in a base and the number of linear units in the height.

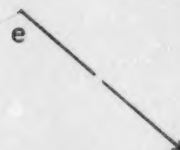
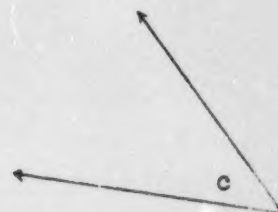
13-11. Cumulative Review.

1. Write the set of factors of 49.
2. Suppose I multiply 2 numbers together and get an answer of 0. What can you say about the numbers I multiplied?
3. Write DCXL in decimal numerals.
4. Mark true or false:
 - a. Any number can be used to build all the counting numbers by addition.
 - b. The number 23 is a prime number.
 - c. The greatest common factor for any two even numbers is at least 2.
5. Write decimal fractions for each of the following. Place a horizontal bar over repeating digits.

a. $\frac{7}{18}$

b. $\frac{1}{14}$

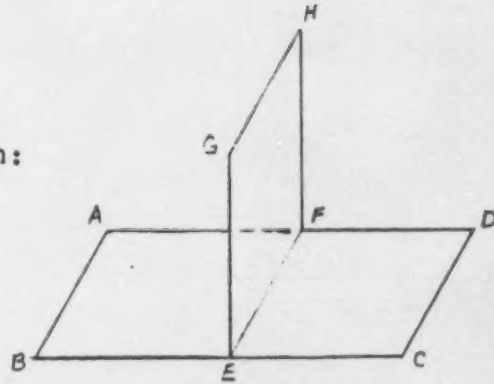
6.



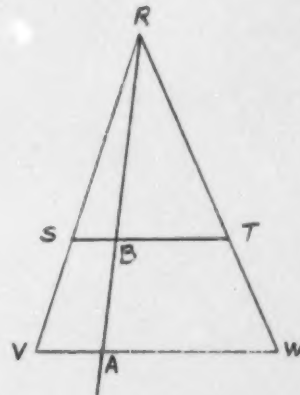
Which of the above angles represent(s) obtuse angle(s)?

7. In the diagram at the right explain how each of the following acts as a separation:

- Plane GFH
- Point F
- $\overleftrightarrow{EF}$

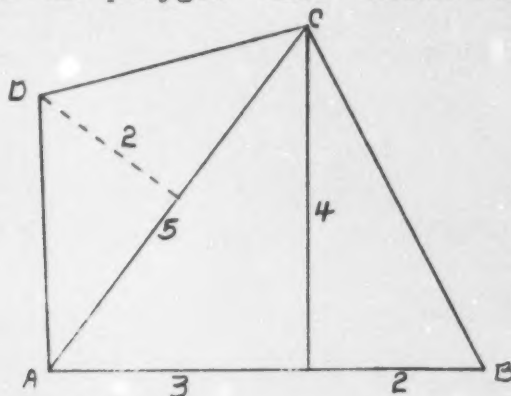


8. By drawing lines through point R (as shown) show how to set up a one-to-one correspondence between the set of points on $\overline{ST}$ and the set of points on $\overline{VW}$.

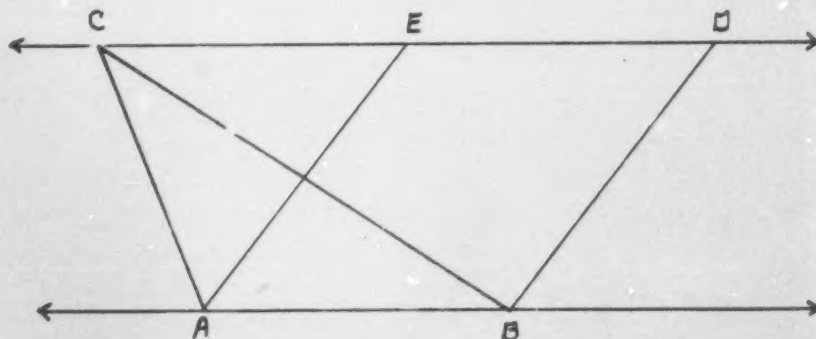


- Find the number of cubic inches in a paper bag that is 1 foot long, 7 inches wide, and 4 inches high.
- Complete: The measure of an angle formed by two perpendicular rays is _____.
- State which of the following ratios are equal:
 - $\frac{68}{17}$, $\frac{760}{19}$
 - $\frac{12}{13}$, $\frac{8}{9}$
 - $\frac{19}{8}$, $\frac{37}{17}$
- Perform the following divisions:
 - $.0081 \div 30$
 - $1.4048 \div .08$

13. List the set of numbers which are less than 100 and also multiples of both 3 and 7.
14. According to the budget in the newspaper, the Mills family should not spend more than 27% of its monthly income on housing. The monthly income of the family is \$390. What is the limit they must set on housing cost per month?
15. A trunk is shaped like a rectangular solid. Find the surface area if the length is $5\frac{1}{4}$ ft., the width is $2\frac{3}{4}$ ft., and the height is $2\frac{2}{3}$ ft.
16. Find the area of the polygon ABCD shown below.



17. How does the area of triangle ABC compare with the area of parallelogram ABDE?



Chapter 14

CIRCLES

Introduction.

The circle is an important geometric figure. It is one of the most common and useful simple closed curves that we know. Transportation was made easier for early man by the invention of the wheel. You are familiar with its use today on bicycles, automobiles, planes, ships, and trains.

It is likely that you have heard the phrase "wheels of progress". This is a tribute to the importance of the circle as a key figure in design and construction.

Cake-tins, pans, plates, guns, engine parts, clock faces, coins, and many other objects suggest circles. Find some in your classroom now and find as many as you can in your home.

You have already talked about simple closed curves. You have studied some properties of such curves as parallelograms, rectangles and triangles. You will now study some of the properties of a circle as a mathematical figure. You know that a simple closed curve may be thought of as a set of points. Let us see how we can describe a circle as a set of points.

14-1. Circles and the Compass.

Choose a point on your paper and label it point P. Now use your ruler to mark at least ten points, each at a distance of 3 inches from P. Do these points suggest any figure? If you answered yes, then what figure? If your points are located in widely different directions from P, they should suggest a circle.

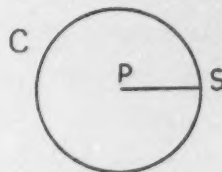
To draw the complete circle, we will use a device called a compass. To draw the circle with a compass, adjust the arms of the compass so that the distance between the sharp point and the pencil tip is 3 inches.

Grasp the handle between the thumb and forefinger. Place the sharp point of the compass at the point P. Then move the compass about this point, making the pencil tip draw a curve.

When it returns to the starting point you have a circle. This exercise should make it clear that a circle is a simple closed curve since you drew it in the manner described in Chapter 4 for drawing a simple closed curve.



Point P is the center of the circle you have drawn. Choose one of the points you marked at a distance 3 inches from P and label it point S . Draw the segment $\overline{PS}$. (Except for size your figure should look like the one below.) We can name this circle in two ways. It can be called "circle P ", meaning a circle which has as center point P or we could name the curve by a capital letter as "circle C ".



The segment $\overline{PS}$ is called a radius of the circle. A radius is any line segment which joins the center P to a point on the circle. Draw another radius of your circle. Call this radius $\overline{PT}$. When you talk about more than one radius say "radii." $\overline{PS}$ and $\overline{PT}$ are radii of the circle. Can you draw other radii of this circle? How many radii may there be of this circle? Of any circle?

Often the word radius is used to mean the distance from the center to any point on the circle. The radius of the circle that you drew is 3 inches. There is only one distance which is "the" radius of a circle, but "a" radius may be any line segment having the center and a point of the circle as its endpoints.

You can now describe a circle as a set of points. The circle is the curve made up of a set of points in a plane. Each point of the circle is the same distance from a certain point called the center. The center is not in the set of points that make up the circle. The circle with center at P and a radius

of r units is the set of all points in the plane at distance r units from P .

Exercises 14-1a

Class Discussion.

1. Look at your drawing again. Choose any new point in the plane and call it point Q . Draw the ray $\overrightarrow{PQ}$. On the ray $\overrightarrow{PQ}$ mark a distance of 3 inches from P . The point you locate will be on the circle. Call the point T . Is $\overline{PT}$ a radius of the circle?
2. a. Label a point O on your paper. Draw a circle with center O and a radius of $2\frac{1}{2}$ inches. Call this circle Y .
b. Choose a point K on the circle. Draw radius $\overline{OK}$. How long is $\overline{OK}$? How long will any radius drawn in this circle be?
3. a. Draw a circle with a 2 inch radius. Label the center of the circle P .
b. Locate 4 points on the circle. Label them A , B , C , and D .
c. Draw the radii $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, and $\overline{PD}$.
d. Do the four radii have the same length? What is the length?

The compass has another use. It is used for transferring distances. Draw a line ℓ and label two points M and N on ℓ . Draw a segment $\overline{CD}$ anywhere on your paper. Without using a ruler we wish to find a point Q on $\overline{MN}$ such that the length of $\overline{MQ}$ is the length of $\overline{CD}$.

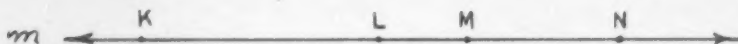


Adjust the arms of your compass so that when you put the sharp point at C , the pencil tip is at D . Then without changing the opening, place the sharp point of the compass at M . Draw a small part of a circle crossing the ray $\overrightarrow{MN}$. Call the point of intersection Q . Then the length of $\overline{MQ}$ is the length of $\overline{CD}$.

Exercises 14-1b

The problems below will give you practice in using your compass to draw circles and to transfer distances. Read the directions carefully, and label each point, circle or line segment before you go on to the next direction.

1. a. Label a point X on your paper. Draw a circle with center X and radius 2 inches. Call it circle G .
 b. Label a point Q of circle G . Draw a circle with center Q and radius 1 inch. Draw another circle with center X and radius 1 inch.
 c. What does the intersection of the last two circles seem to be?
2. In this problem, you are to transfer some distances to your paper from the figure below.



- a. Draw a horizontal line ℓ on your paper, and label a point P on the line near the right arrowhead.
- b. Use your compass to locate point Q to the left of P on line ℓ , so that $PQ = KL$ (PQ with no symbol above it means the measure of segment $\overline{PQ}$).
- c. Locate point R to the left of P on line ℓ , so that $PR = KM$.
- d. Locate point S to the left of P on line ℓ , so that $PS = KN$.
- e. With Q as center and QS as radius, draw a circle.
- f. If your drawing is accurate, there will be two labeled points on the circle. Name the points.

3.
 - a. Draw two intersecting lines which are not perpendicular. Call the lines ℓ and m .
 - b. Label the point of intersection of the two lines point E.
 - c. With E as center and radius 1 inch, draw a circle.
 - d. Label the intersection of the circle with line ℓ points F and G.
 - e. Label the intersection of the circle with line m points H and K.
 - f. Draw $\overline{HF}$, $\overline{FK}$, $\overline{KG}$, and $\overline{GH}$.
 - g. What kind of figure does HFKG seem to be?
- *4. Draw a line ℓ and label points X and Y 1 inch apart on ℓ .
 - a. Draw a circle A which has its center at X and passes through Y.
 - b. Draw a circle B which has its center at Y and passes through X.
 - c. Label Z as the other intersection of circle B and line ℓ .
 - d. Draw a circle C which has its center at Z and passes through X.
 - e. What is the intersection set of circle A and circle C?
 - f. What is the intersection of circle B and circle C?

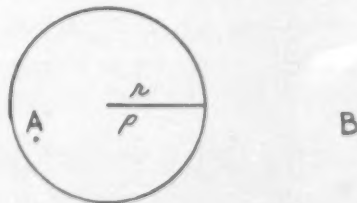
14-2. Interiors and Intersections.

A circle is a simple closed curve. A simple closed curve has an interior and an exterior. Then a circle has an interior and an exterior. When is a point in the interior of a circle and when is a point in the exterior? The exercises below will help you answer this question.

Exercises 14-2a

1. Is the center of a circle in the interior or the exterior of the circle?

2. Draw a circle with center Q and a radius of 2 inches. Mark a location one inch from Q . Label this point A . Is the point in the interior or exterior of the circle?
3. Mark a location $1\frac{1}{2}$ inches from Q . Label this point B . Where is B ?
4. Mark four other points inside the circle and label them C , D , E , and F . How far is each of these points from Q ?
5. Compare the distances $\overline{QA}$, $\overline{QB}$, $\overline{QC}$, $\overline{QD}$, $\overline{QE}$, and $\overline{QF}$ with the radius of the circle Q .
6. Complete the statement: A point is in the interior of a circle if its distance from the center of the circle is less than the _____.
7. Locate a point S outside the circle in Problem 2 above. How does $\overline{QS}$ compare with the radius of the circle?
8. Locate a point H outside the circle in Problem 2 above. How does $\overline{QH}$ compare with the radius of the circle?
9. Locate a point K so that $\overline{QK} > 2$.
10. Complete the statement: A point is in the exterior of a circle if its distance from the center of the circle is _____ the radius.
11. Draw a circle with center P and a radius r units. Here we mean P is any point and r is any convenient radius.
12. Locate a point inside the circle like A in the figure below. The point A is _____ than _____ units from the center P .
13. Locate another point inside the circle. Label this point C . What can you say about the distance PC ?
14. Complete the statement: The interior of a circle of radius r is the set of all points at a distance _____ than _____ units from the _____ of the circle.



15. Locate a point outside the circle like B. The point B is _____ than _____ units from the center P.
16. Locate another point outside the circle. Label this point D. What can you say about the distance PD?
17. Complete the statement: The exterior of a circle of radius r is the set of all points at a distance _____ than _____ units from the _____ of the circle.

In Chapter 4 you worked with notions of intersection and union of two sets. Since the circle is an example of a set of points, you may raise questions about intersections and unions involving circles. You may recall that the intersection of two sets is the set that contains all of the elements that are common to the two given sets. Let us review this idea by looking at two sets of numbers.

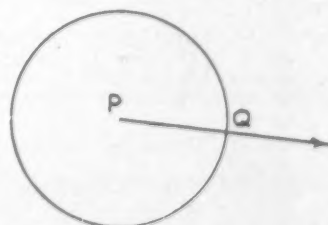
$$A = \{1, 3, 5, 7, 8, 9\} \quad B = \{1, 2, 3, 4, 5, 6, 7\}$$

What is the intersection of set A and set B? Call this set C. Then $C = \{1, 3, 5, 7\}$. Set C contains all of the elements that are in set A and also in set B. We use the symbol $\cap$ to mean intersection, and we write $A \cap B = C$.

Exercises 14-2b

Class Discussion.

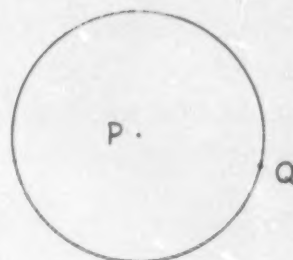
1. Choose a point on your paper. Label it P. Choose any length and adjust the arms of your compass to this length. With P as center and the length you chose as the radius, draw a circle. Choose a point Q on the circle. Draw the ray $\overrightarrow{PQ}$. Your drawing should look like the one on the right.
 - a. Are any points of the circle also on the ray $\overrightarrow{PQ}$? Name any you find.



- b. Name the intersection of the circle P and the ray $\overrightarrow{PQ}$ by completing this statement:

$$\text{Circle } P \cap \overrightarrow{PQ} =$$

2. In the same circle draw $\overrightarrow{QP}$.
- Describe the intersection of circle P and $\overrightarrow{QP}$.
 - How many points belong to this intersection set?
3. Draw a circle with center M and a radius of $1\frac{1}{2}$ inches.
- Describe the set of points in the interior of circle M .
 - Describe the set of points on circle M .
 - Describe the set of points in the exterior of circle M .
4. a. In the drawing at the right, is any point of the circle also in the interior of the circle?
- b. Is any point in the interior of the circle also on the circle?
- c. Complete the following statement:
The intersection of circle P and the interior of circle P is _____.



You can see that no point of the circle is also a point of the interior of the circle. We say that the intersection of a circle and its interior is the empty set.

Look at the figure above which consists of the circle and its interior. What is the union of the circle and its interior? (Did you remember that the union of two sets is the set containing all of the elements that are in the first set or the second set (or both)?)

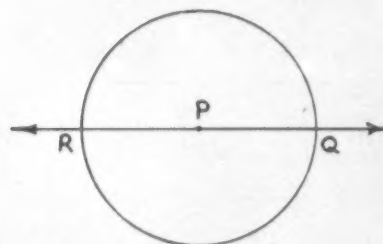
The set of all points which are either on the curve or inside the curve is the union of the circle and its interior. We can say the union of a circle and its interior is the set of all points whose distance from the center P is either the same as or less than the radius of the circle.

Exercises 14-2c

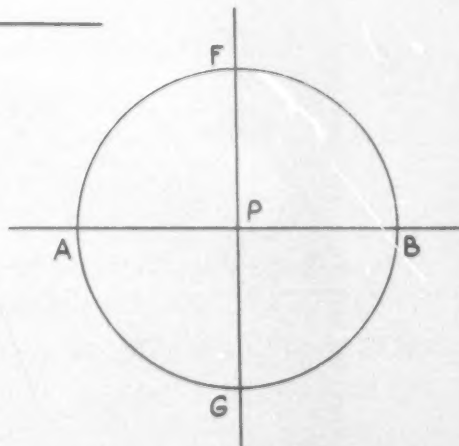
Class Discussion.

Let us use the letter D to represent the union of our circle and its interior.

1. What is the intersection of the set D and the line $\overleftrightarrow{PQ}$?
2. In what way is the set $D \cap \overleftrightarrow{PQ}$ quite different from the intersection of $\overleftrightarrow{PQ}$ and the circle P ?
3. Locate two points. Label them P and Q . Draw two circles with center at P such that Q is in the exterior of one circle and in the interior of the other. Label the first circle G and the second circle D .
4. Let C be a circle with center at P and radius r units. Let S be any other point in the same plane. (Draw the picture.)
 - a. How many points belong to the intersection of the circle C and the ray $\overrightarrow{PS}$?
 - b. How many points belong to the set $C \cap \overleftrightarrow{PS}$?
 - c. Do your answers to parts (a) and (b) depend upon where you chose the point S ?
 - d. How many points belong to the set $C \cap \overleftrightarrow{PS}$? Does your answer depend on the choice of S ? Explain.



In the figure to the right, the point P is the center of the circle. The four points A , F , B , G lie on the circle. The intersection of the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{FG}$ consists of the point P , and the two lines are perpendicular. Copy this diagram on a piece of paper. (The steps in copying are as follows: with compass draw



the circle, mark the center P . Draw the line AB through P . Use a protractor to obtain a right angle, draw FG perpendicular to $\overleftrightarrow{AB}$. Label the points.)

Exercises 14-2d

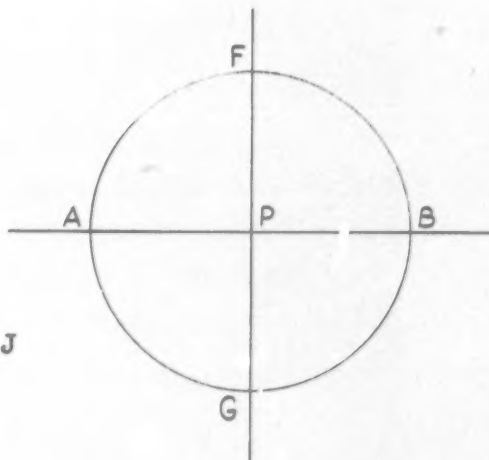
Class Discussion.

1. In your drawing, shade the halfplane which contains F and whose boundary is the line $\overleftrightarrow{AB}$. Call this halfplane H . (A colored pencil or crayon is good for shading. If you do not have one, shade lightly with your ordinary pencil.)
2.
 - a. What is the intersection of the halfplane H and the circle?
 - b. Does A belong to this intersection? Does G ? Does F ? Does P ?
 - c. Can you count how many points belong to this intersection?
3.
 - a. Choose two points of the intersection above which also belong to the interior of the angle BPF . Label these points M and N .
 - b. Choose a point K which lies on the circle and also in the interior of the angle APF .
 - c. Draw the angle MKN . What is the intersection of the circle and the angle MKN ?
 - d. What is the intersection of the interior of the circle and the interior of the angle MKN ?

Exercises 14-2e

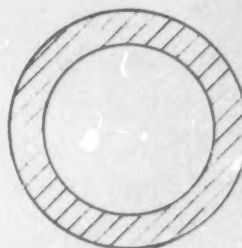
1. In a plane, can there be two circles whose intersection consists of just one point?
2. Draw a circle. Then draw a straight line that intersects the circle. Can more than two points belong to the intersection of the line and the circle?

3. Draw a line segment $\overline{AB}$. Locate some point P on $\overline{AB}$. With A as center and $\overline{AP}$ as radius draw circle A . With B as center and BP as radius, draw circle B . What is the intersection of the two circles?
4. Locate two different points P and Q . Draw the circle with center at P and with segment $\overline{PQ}$ as a radius. Then draw the circle with center at Q and with P on the circle.
 - a. What is the intersection of these two circles? Label other points if necessary.
 - b. Can you draw a line which passes through every point of the intersection of the two circles? Can you draw more than one such line? Why?
 - c. In your picture shade the intersection of the interiors of the two circles. (If you have a colored pencil handy use it for shading; if you do not, use your ordinary pencil and shade lightly.)
 - d. Shade the intersection of the interior of the circle with center P and the exterior of the circle with center Q . (Use a different type of shading from that in (c) above. Did you find it helpful to mark separately the two sets whose intersection you needed before doing the shading?)
 - e. Make another drawing of the two circles. On it shade the union of the interiors of the two circles.
5. The diagram at the right is one you copied in class. We let the halfplane on the F -side of $\overleftrightarrow{AB}$ be called H . Let J be the halfplane on the B -side of $\overleftrightarrow{FG}$.
 - a. What is the set $H \cap J$?
 - b. What is the intersection of all three of the sets J and H and the circle?



- c. Does the diagram suggest to you that the circle has been separated into what might be called quarters? If so, can you describe several of these portions?
- d. Can you find a portion which might be called half of the circle? Can you describe it in the language of intersections or unions of sets? Can you identify more than two such portions?

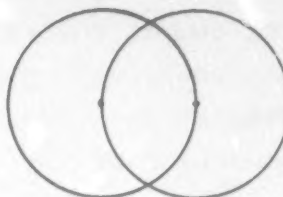
6. In the figure to the right, the two circles are concentric. That means they lie in the same plane and have the same center.



- a. Describe the intersection of the two circles.
- b. Describe the shaded region. (Use such words as "intersection," "interior," "exterior".)

- *7. Look at the figure in Exercise 6. How can you describe the intersection of the exteriors of the two circles in a very simple way?

- *8. The center of each circle lies on the other circle. Copy the figure on your paper. Shade the union of the exteriors of the two circles.

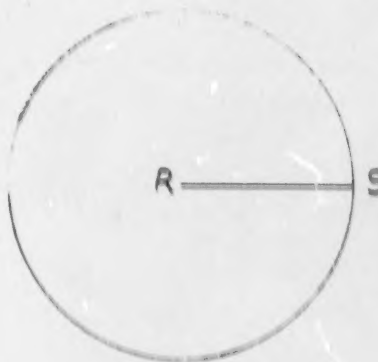


14-3. Diameters and Tangents.

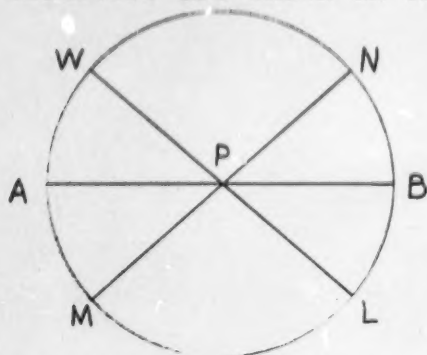
The meaning of the word "radius" is closely connected with the meaning of the word "diameter." We use the word "radius" in two different ways. We say the radius of a circle is one of the segments joining a point on the circle and the center. We also say that the radius of a circle is the length of any one of

these segments. For example, consider the circle below. We can say that a radius of the circle is $\overline{RS}$. Here we use the word radius to mean the segment, $\overline{RS}$.

We can also say that the radius of the circle is one inch. Here we use the word radius to mean the length of the segment $\overline{RS}$.



A diameter of a circle is a line segment which contains the center of the circle and whose endpoints are on the circle. A circle has many radii. It also has many diameters. Three diameters are shown in the circle below.



The diameters are $\overline{AB}$, $\overline{MN}$, and $\overline{WL}$. How many radii are shown in the figure?

A diameter is a set of points. A set of points which is a diameter may be described in another way. A diameter of a circle is the union of two different radii which are segments of the same line. In the figure above, the diameter $\overline{AB}$ is the union of the two radii $\overline{AP}$ and $\overline{PB}$. Describe the diameter $\overline{MN}$ as the union of two radii. How does the length of a diameter compare with the length of a radius in the same circle?

We use the word diameter in two ways just as we use the word radius. Sometimes we will say "diameter" to mean a particular segment and sometimes we will say "diameter" to mean the length of a segment. For example, consider the circle above. We can say $\overline{AB}$ is a diameter of the circle. We can also say the diameter of the circle is 2 inches. We can usually tell from the problem which meaning to give to the words radius and diameter.

Exercises 14-3a

Class Discussion.

1. Draw a circle with radius 3 inches. How long is the diameter of the circle?
2. Draw a circle with a diameter of 4 inches. How long is the radius of the circle?
3. Do you believe that the following statement is true? "The measure of the diameter is twice the measure of the radius."

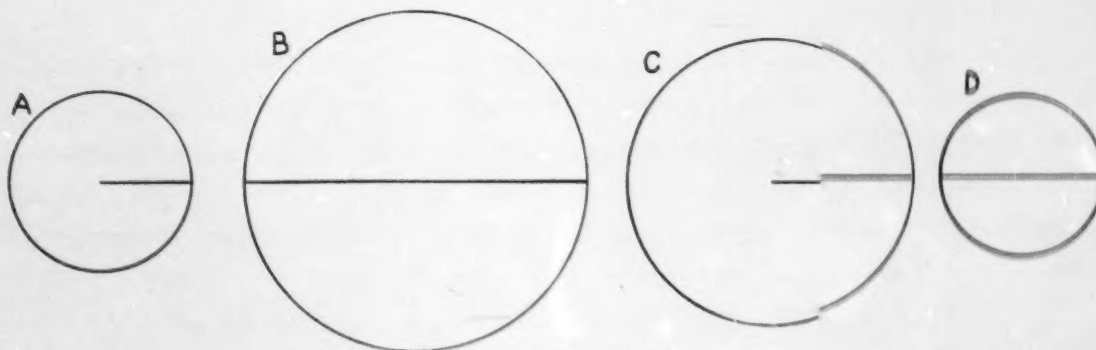
If we choose any unit of length and if we let r be the measure of the radius and d the measure of the diameter (of the same circle), then we have the important relationship:

$$d = 2 \cdot r \quad \text{or} \quad r = \frac{1}{2} \cdot d.$$

Exercises 14-3b

Class Discussion.

Using the relationship above, give the missing information for the circles below.

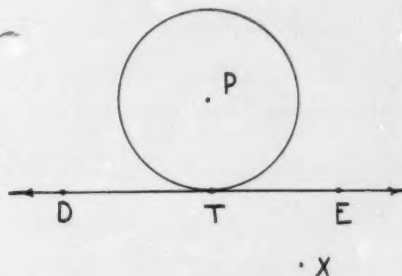


1. In circle A, if $r = 3$, then $d = \underline{\quad ? \quad}$.
2. In circle B, if $d = 10$, then $r = \underline{\quad ? \quad}$.
3. In circle C, if $r = 4\frac{1}{2}$, then $d = \underline{\quad ? \quad}$.
4. In circle D, if $d = 3$, then $r = \underline{\quad ? \quad}$.

Exercises 14-3c

1. Draw a circle. Locate a point on the circle. Call the point R.
 - a. How many radii of the circle contain the given point R?
 - b. How many diameters of the circle contain the given point R?
 2. The diameter of a circle is 18 inches. How many inches from the center of the circle is a point on the circle?
 3. Draw a circle C with center at the point P. Draw three diameters of C. Draw a circle with center at P whose radius is equal to the diameter of C.
 4. (You must handle your compass very carefully in order to do this problem.)
 - a. Near the middle of a sheet of paper, mark a point Q.
 - b. Draw a large circle C with center at Q. Do not change the opening of the compass arms after the drawing is completed.
 - c. Mark a point U on the circle C.
 - d. With the compass, find a point V on the circle C such that $\overline{UV}$ has the same length as $\overline{QU}$.
 - e. Again with the compass, find a third point W on C such that $\overline{VW}$ has the same length as $\overline{QU}$.
 - f. Continue on around, locating points X, Y, Z on C, (Use your compass three times.) such that the length of each of the segments $\overline{WX}$, $\overline{XY}$, $\overline{YZ}$ is the same as the radius of the circle.
 - g. Compare the length of the segment $\overline{ZU}$ with the radius of the circle.
 - h. What name could you give to each of the segments $\overline{UX}$, $\overline{VY}$, and $\overline{WZ}$? (If your drawing was very carefully done the simple closed curve UNWXYZ would represent a "regular hexagon.")
-

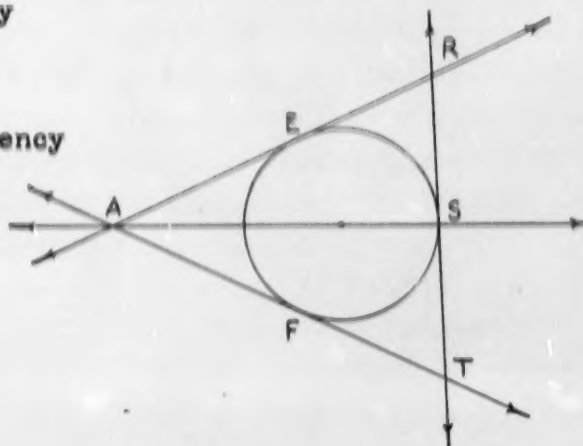
The line and the circle in the figure are similar to a wheel resting on a track. How many points on the circle are also on the line? Name each such point. We may say that the line is tangent to the circle. The single point T of their intersection may be called the point of tangency.



Exercises 14-3d

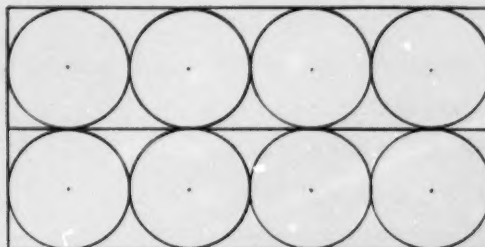
Class Discussion.

1. In the figure above, the line DE separates the plane DPX into two halfplanes.
 - a. How can you describe the intersection of the halfplane containing X and the circle?
 - b. What is the intersection of the halfplane containing P and the interior of the circle?
2. Can you draw a circle and a line in the same plane such that their intersection is the empty set?
3. Can you draw a circle and a line such that their intersection contains four points.
4. How many lines are shown in the figure? How many of these lines are not tangent to the circle? Name each point of tangency shown.



Exercises 14-3e

1. How many illustrations of the tangent notion do you find in this pattern?



2. The diameter of a certain circle is 42 cm. How many centimeters from the center of the circle is a point on the circle?
3. In the diagram the point Q is the center of the circle and the center of the square $EFGH$.

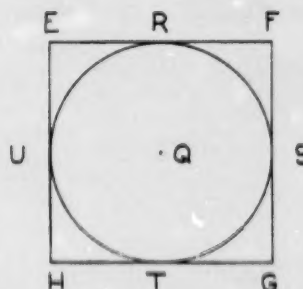
a. What is the intersection of the circle and the square?

b. What is the intersection of the line $\overleftrightarrow{GH}$ and the circle?

c. What new name have we given to the point T ?

d. How many other lines (segments of which are represented in the figure) are tangent to the circle?

e. Name all the points of tangency you can.



4. On your paper make a sketch of the diagram for Problem 3. (A careful drawing is not needed here.) On your copy, draw the quadrilateral $RSTU$.
- a. How many sides of this quadrilateral are segments of lines tangent to the circle?
- b. What is the intersection of the interior of the circle and the exterior of the square $EFGH$?
- c. Describe carefully the intersection of the exterior of the circle and the interior of the square $EFGH$.
- d. What is the intersection of the interior of the circle and the exterior of the quadrilateral $RSTU$?

- Refer again to the carefully drawn figures in Problem 3.
- *5.
- Do you think that all three of the points R , Q , T lie on one line?
 - Do you believe that the three points U , Q , S lie on one line?
 - Make an estimate of the size of the angle QTQ . Can you check your estimate with a protractor?
 - Make an observation about a relationship between the line QS and the line FG . Can you check your observation with a protractor?
- *6. Draw any circle and any line tangent to the circle. Draw also the line which joins the center of the circle and the point of tangency. Do you believe there is an important relationship between these two lines? If so, what is it?
- *7. We know from the description of a circle that all radii have the same length. How do we know that all diameters of a given circle have the same length?

14-4: Angle and Central Angles:

In Chapter 7 you learned that a single point on a line separates the line into two half-lines. This idea of separation led you to see that on a line a single point determines two half-lines.

Exercises 14-4a

Class Discussion:



Figure 1

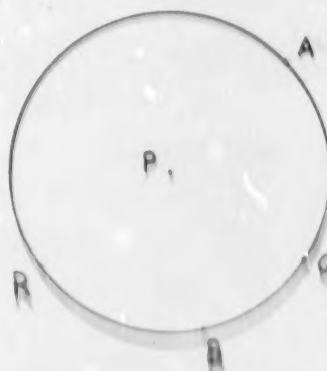


Figure 2

Study the figures above.

1. a. In figure 1, does the single point Q separate the whole circle into two parts?
b. How many points do we need to separate a circle into two parts?
2. In figure 2, do points X and Y together separate the circle into two parts?

In the figure to the right, the two points A and B separate the circle into two parts. One part contains the point R and the other contains the point Q. There is no path along the circle from R to Q that does not include at least one of the points A or B.



It takes two different points to separate a circle into two different parts.

In the drawings below, A, Q, and B are points on a line.



Q would be considered "between" A and B in each drawing. This is tied in with the separation properties of a point on a line.

Let us consider a different situation.



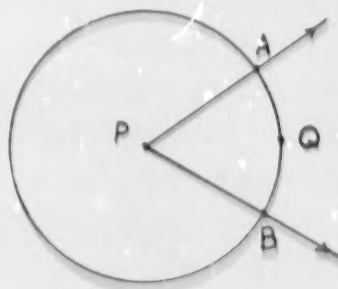
Many observations could be made from the figure above. On the circle, Q could be considered between A and B. Suppose a clockwise path is traced from Q around to A. We could say that B is between Q and A. Can you look at the diagram and

think of A as being between B and Q .

On a line "betweenness" and "separation" are closely related. On a circle, or other simple closed curve, a single point does not separate the curve into two parts.

In earlier chapters on geometry, we talked about segments of lines. It will often be necessary for us to consider portions of a circle, or arcs.

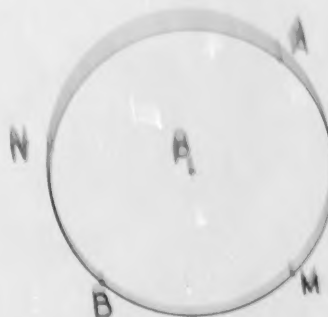
In the figure to the right the points A and B separate the circle with center at P into two portions. Each of these pieces together with A and B is an arc. A and B are called end-points of the arc. Notice that any two different points on a circle



determine two different arcs. Both arcs have the same two points A and B as end-points. It is often hard to tell which of these arcs we are talking about when we look at this picture. Therefore, we sometimes label a point on the circle between the end-points like Q . Then the small arc in the drawing may be read as $\widehat{AQB}$.

Exercise 14-4b

In the drawing to the right, points M and N have been labeled between the end-points of the arc AB . Use the symbol $\widehat{AB}$ to represent an arc whose end-points are A and B . In Exercises 1-3 below we are interested in the arcs whose end-points are A and B .

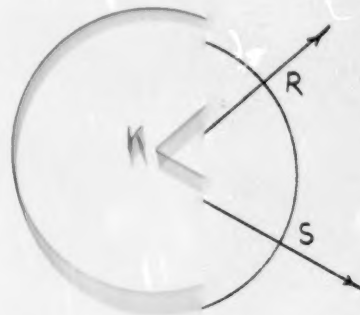


1. Use symbols to represent the arc containing the point M .
2. How would you represent the arc containing the point N .

- 3: Use symbols and describe the two areas above another way.
- 4: Draw a circle. Draw a diameter of the circle and label its end-points A and B.
- a: Tell how points A and B separate the circle.
- b: What name is given to the area into which A and B have separated the circle?

Whenever the end-points of an arc are also end-points of a diameter, the two arcs are semi circles.

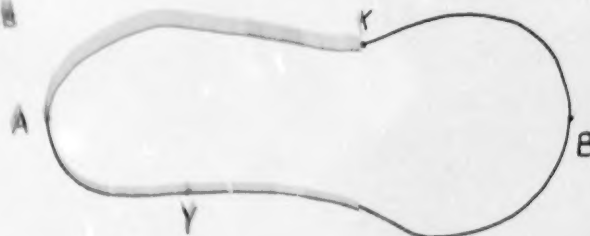
Draw a circle. Label the center K. Choose any two points on the circle that are not end-points of a diameter. Label these points R and S. Draw the rays $\overrightarrow{KR}$ and $\overrightarrow{KS}$. Notice $\angle RKS$. The vertex of the angle is at K, the center of the circle. The sides of the angle are the rays $\overrightarrow{KR}$ and $\overrightarrow{KS}$. Since the vertex of the angle is at the center of the circle, $\angle RKS$ is called a central angle.



Exercises 14-4

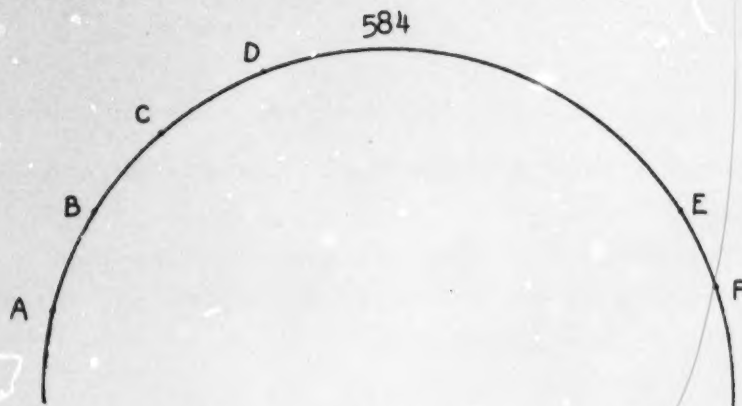
- 1: Notice the drawing to the right. The pair of points A and B separate the points X and Y. Which points, if any, do the pairs of points below separate?

- a. A, Y
b. X, Y
c. B, X



14-5

2.

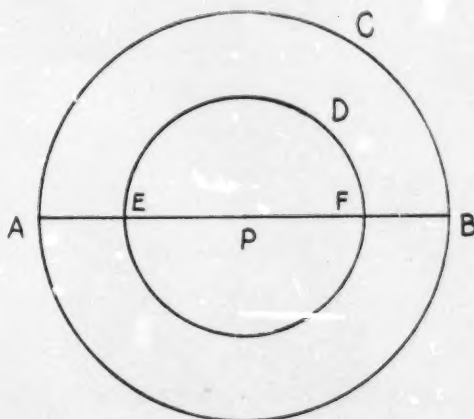


Given the arc $\widehat{ABCDEF}$ above, determine the following:

- | | |
|-------------------------------------|-------------------------------------|
| a. $\widehat{AC} \cap \widehat{BD}$ | d. $\widehat{CD} \cap \widehat{DE}$ |
| b. $\widehat{AF} \cap \widehat{DF}$ | e. $\widehat{DF} \cap \widehat{AE}$ |
| c. $\widehat{AD} \cap \widehat{CF}$ | f. $\widehat{AB} \cap \widehat{BF}$ |

14-5. Length of a Circle.

Look at the two circles in the drawing. Circle C and circle D have the same center P. Name the diameter of circle C; of circle D. Which circle has the longer diameter? How do you know? Which circle seems to be longer? It looks as if the longer circle has the longer diameter. Perhaps there is some connection between the length of a circle and the length of its diameter. The length of a circle is called its circumference.



Let us see what we can discover about the circumference and diameter of a circle. We will try two experiments and then compare the results.

Exercises 14-5a

1. Experiment 1. (Do this very carefully at home and bring your results to class.)
 - a. Choose three circular objects in your home. Use a tape measure to find the length of each circle. (If you do not have a tape measure, use a piece of string and then measure the string.)
 - b. Measure the diameters of the same three circular objects. It may be hard to find the center of your circle, so measure the diameter several times to get the best measure.
 - c. Make a table like the one below and put the results of your experiment in the table.

NAME OF OBJECT	MEASURE _c OF CIRCLE	MEASURE _d OF DIAM.	$c - d$	$\frac{c}{d}$
1. WATER GLASS	—	—	—	—
2.				
3.				

- d. To compare two quantities, we find their difference, or find their ratio. Find the difference between the circumference and the diameter of each of your circular objects. Write the results in the column headed " $c - d$ ". Do the differences seem to be the same?
- e. Find the ratios by dividing the measure of the length of each circle by the measure of its diameter. Write the quotient in the column headed " $\frac{c}{d}$ ". (Round your quotients to the nearest tenth.) Do the ratios seem to be about the same for all three circles?
- f. The results in the fourth column should suggest that the circumference of each circle is about ? times as long as its diameter.
- g. Compare the ratios in your $\frac{c}{d}$ column with those of your classmates. What seems to be true about them?

2. Experiment 2. Do part (a) of this experiment at home.

The rest should be a class exercise.

- a. On stiff paper or cardboard mark a point P . Draw a circle with center P and diameter between 2 inches and $2\frac{1}{2}$ inches. Now cut along the circle so that you have a figure with the circle you drew as its boundary.
- b. Draw a line ℓ about 10 inches long. Label a point A near the left end of the line. Now locate point B on ℓ so that the segment $\overline{AB}$ has the same length as the diameter.
- c. Make a number scale on line ℓ with 0 at point A and 1 at point B . Continue the scale to the number 4 or 5.
- d. Mark a point on your circle and label it C . Place the circle so that it is tangent to your number line, with C at the zero point.
- e. Carefully roll the circle to the right along the number line. Each point on the circle will touch a point on the line. Continue rolling the circle until point C is again on the line. Label the point where C touches the line point D .
- f. The point D is between what whole numbers? What decimal number do you think corresponds to the point D on the number line?
- g. When you rolled circle P along the number line, each point on the circle touched exactly one point of the line. No two points of the circle touched the same point of the line. What segment on the number line has the same length as the circle?
- h. What segment on the number line has the same length as the diameter of the circle?
- i. How does the length of the circle seem to compare with the length of its diameter?

Does the result of Experiment 2 agree with your results in Experiment 1?

Do the results of these two experiments suggest the following statements?

1. For any circle, the ratio of the length of the circle to the length of its diameter is always the same number.
2. This number is a little more than 3.
3. If the experiments are carefully done, the results suggest that this number is between 3.1 and 3.2.

Mathematicians have proved that the first statement above is correct. A special symbol is used for the number which is the ratio of the length of a circle to the length of its diameter. The symbol is written " π " and is read "pi". (It is a letter from the Greek alphabet.)

We could shorten the first statement by saying $\frac{c}{d} = \pi$ where c is the measure of the circle and d is the measure of its diameter. We can also write the statement $c = \pi \cdot d$.

Earlier in the chapter we noticed the important relationship of the diameter and the radius of a circle. Using this fact, can we write a statement about the length of a circle and the length of its radius?

We know $d = 2 \cdot r$ and $c = \pi \cdot d$

then $c = \pi \cdot (2 \cdot r)$ (Using the associative property for multiplication)
or $c = (\pi \cdot 2) \cdot r$

or $c = (2 \cdot \pi) \cdot r$ (Using the commutative property for multiplication)

or $c = 2\pi r$

The Number π

The number which the symbol " π " represents is a new kind of number. It is not a whole number. It is not a rational number. You will learn more about this kind of number when you take later courses in mathematics. For our work in this chapter we will agree to use 3.14 or $\frac{22}{7}$ to approximate the number π .

1

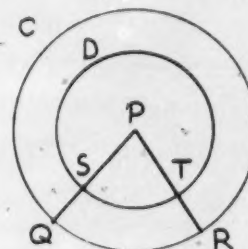
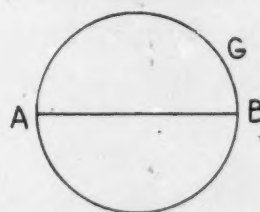
Class Exercises 14-5b

1. See how close 3.14 and $\frac{22}{7}$ are to each other.
 - a. Multiply each by 100. Examine your results. Are your answers close to the same whole number?
 - b. Divide 22 by 7. Carry the division out to three decimal places. Did you get an answer very close to 3.14?
2. What is the circumference of a circular table whose diameter is 7 feet?
3. How much fencing is needed to inclose a circular flower garden if the radius of the circle is 15 feet?
4. Since c is the product of the factors π and d , complete the following statement using only the symbols c , π , and d .
 $d =$
5. c is also the product of the factors $(2 \cdot \pi)$ and r . Use these symbols to complete the following statement.
 $r =$
6. A wheel moves a distance of 12 feet along a track when the wheel turns once. What is the diameter of the wheel?

Exercises 14-5c

1. Using $\pi \approx 3.14$, find the missing information about the circles described below.
 - a. Circle A: Radius ? Diameter 10 inches
Circumference ?
 - b. Circle B: Radius 4.2 cm. Diameter ?
Circumference ?
 - c. Circle C: Radius ? Diameter ?
Circumference 25 feet
 - d. Circle D: Radius ? Diameter 12.5 inches
Circumference ?

2. Using $\pi \approx \frac{22}{7}$, find the following.
- The length of a circle whose diameter is 14 inches.
 - The length of a circle with radius 21 feet.
 - The diameter of a circle with circumference 132 inches.
 - The radius of a circle with circumference 44 feet.
 - The circumference of a circle with radius $10\frac{1}{2}$ inches.
3. The diameter of a circular tablecloth is 35 inches. What is the circumference of the tablecloth?
4. A circular lampshade 12 inches in diameter needs new binding around the bottom. How long a strip of binding will be needed?
5. The circular track at Central High School has a radius of 110 feet. What is its circumference?
6. A strip of metal 62 inches long is to be made into a circular hoop. What will its diameter be?
7. A merry-go-round in a playground has a 15 foot radius. If you sit on the edge, how far do you ride when it turns once?
8. A circle with a diameter of 20 inches is separated by points into 8 arcs of equal length.
- What is the length of the whole circle?
 - What is the length of each arc?
9. In the figure A and B are the end-points of the diameter of circle G. If the diameter of circle G is 8 inches, then what is the length of each semicircle?
10. In the figure, circle C and circle D have the same center P. The radius of circle C is 7 inches and the radius of circle D is 5 inches.
- Find the length of each circle.
 - If arc $\widehat{QR}$ is one-fifth of circle C and arc $\widehat{ST}$ is one-fifth of circle D, then the linear measure of arc $\widehat{QR}$ is _____ and that of arc $\widehat{ST}$ is _____.



14-6. Area of Circles.

Do you have a circular frying pan in your kitchen? Perhaps it is a "nine-inch skillet." The boundary of the frying surface represents a circle. The "nine-inch" tells the diameter of the circle. Have you seen some electric frying pans that are square shaped? You may have both of these pans in your home. When you look at these two pans, you may ask which is bigger. In this case bigger means more frying surface. To answer the question you must compare the area enclosed by the circle with the area enclosed by the square. You may decide that the areas are so close that you cannot tell by sight which pan is bigger. The rest of this section should help you answer questions of this kind.

In an earlier chapter, you learned how to find the area of a closed region. The agreement was that you can say "area of a square" to mean the area of the region enclosed by a square. The boundary of the frying surface in the electric pan represents a square whose sides are eight inches long. Then what can be said about the area of the interior of the pan? Can you say the area of the interior is sixty-four square inches? You have not studied a method of computing the area of a region enclosed by a circle whose diameter is known.

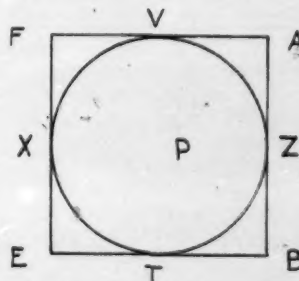
When we talk about a circle, we usually shorten the phrase "the area of the region enclosed by a circle" and say "the area of the circle." Often the area of a circle is expressed in terms of the radius of the same circle.

Exercises 14-6a

Class discussion.

The drawing on the right shows a circle and a square.

1. What is the center of the circle?

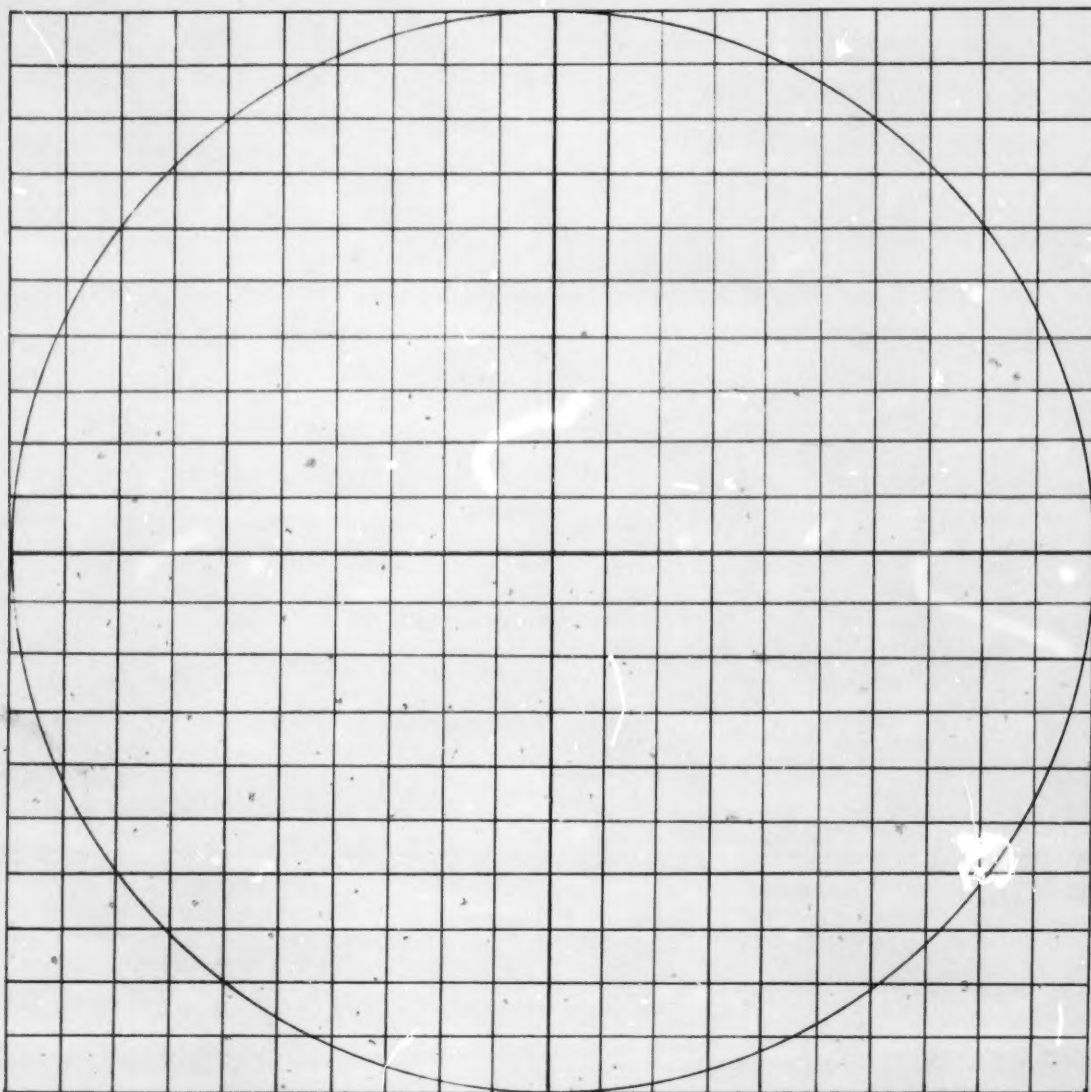


2. Let the measure of the radius of the circle be r . Complete the following statements. Your answers should be in terms of the radius of the circle.
 - a. Each of the segments $\overline{XP}$ and $\overline{PZ}$ is a _____ of the circle.
 - b. The measure of the diameter $\overline{XZ}$ is _____.
 - c. The measure of segment $\overline{EB}$ is _____.
 - d. The measure of segment $\overline{AB}$ is _____.
 - e. Give reasons for your answers in parts (a), (b), (c), and (d).
3. Since each side of square $ABEF$ has a measure of $2 \cdot r$ then the area of the square is _____.
4. Do you think that the area of square $ABEF$ is four times as large as the area of square $AZPV$? Why? (You may draw segments $\overline{PZ}$ and $\overline{VP}$ in the figure.)
5.
 - a. How does the area of the circle compare with the area of square $ABEF$?
 - b. Do you think that a circle whose radius is r units has an area which is less than $4 \cdot r^2$ units?
6. Would you guess that the area of the circle is more than one-half the area of the square? More than two-thirds? More than three-fourths?

Suppose you try to approximate the area of a circle by careful measurement. This will tell you whether or not you made a reasonable guess. It will be a good idea to turn back to the chapter on measurement. Review the basic notion of a unit of measure and how you estimate a size by finding how many units of a given kind are contained in the figure.

The following exercises should help you find a way to approximate the area of a circle. Below are two drawings. The small one shows a square each of whose sides is one unit long. This square represents the unit of area for these exercises. The

large figure shows a circle inside a large square. The interior of the large square has been covered with units of area. Can you find the number of units of area in this circle?



Exercises 14-6b

Class Discussion.

1. How many linear units are there in the radius of the circle?
 2. How many units of area are in one of the squares whose side is the length of the radius of the circle?
 3. How many units of area are there in the interior of the circle? (Several of the units of area are separated into two parts by an arc of the circle. If half or more than half of a unit of area is in the interior of the circle, count it. Do not count less than half of a unit of area in the interior of the circle.)
 4. Divide the number of units of area in the interior of the circle by the number of units of area in the square. What is your quotient?
 5. Does your quotient in Exercise 4 look familiar? It should be close to 3.14. What is the name for this number?
-

Can you make a statement about the area of the interior of the circle based on the exercises that you have just completed? Do you think the following statement is true? The area of the interior of the circle is about π times the area of a square whose side is the length of the radius of the circle?

Exercises 14-6c

- a. On a sheet of rectangular graph paper, draw a large circle. (Use your compass.)
- b. Using the shortest unit of length printed on the paper, measure the radius of the circle.
- c. Count the number of units of area in the interior of the square whose side is the length of the radius of the circle.
- d. Count the number of units of area in the interior of the circle.

- e. Divide the number of units of area in the interior of the circle by the number of units of area in the interior of the square.
- f. Is your answer close to 3.14?

We have found by measurement an estimate of the area of certain circles. We may say the area of a circle is π times the area of a square whose side is the length of the radius of the circle. We can write this statement in symbols as $A = \pi r^2$. (Remember r^2 is the second power of r and means the same as $r \times r$.)

Now we are ready to complete our problem about the frying pans. The nine-inch frying pan has a diameter of 9 inches. Its radius is $\frac{9}{2}$ inches. The number of square-inches in its area is $\pi \cdot (\frac{9}{2})^2$. This is the same as $\frac{81}{4}\pi$ square inches, which is nearly 63.6 square inches. The area of the interior of the nine-inch circular skillet is nearly 63.6 square inches.

Remember the area of the interior of the square frying pan was 64 square inches. Is there enough difference between the two pans to make a problem for the cook?

Exercises 14-6d

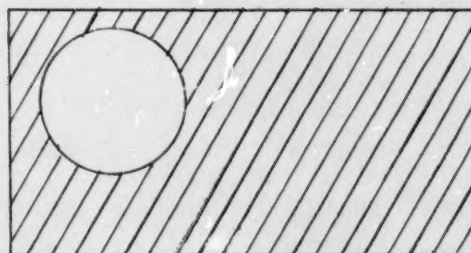
1. A circular wading pool has a radius of 7 feet. What is its area?
2. What is the area of a circle that is 12 feet in diameter?
3. Information is given for four circles. The letters r , d , c , and A are the measures of the radius, the diameter, the length (or circumference), and the area respectively. Find the missing information. Use 3.1 as an approximation to π .

r	d	c	A
20	—	—	1240
—	8	—	—
—	—	0.093	—
12	—	—	—

Note that $\pi \cdot (20)^2 \approx (3.1) \cdot (400)$
 $= 1240$

4. Which has the greater frying surface--an eight-inch circular skillet or a seven-inch square frying pan?
5. A circular drum-head is twelve inches in diameter. What is the area of the drum-head?
6. What method is easier for finding the area of a circle--to measure the radius and calculate the area, with the aid of $\pi \cdot r^2$, or to measure the area directly with an appropriate unit of measure? Should both methods yield the same result?
7.
 - a. Find the area of a circle that has a radius of 2 inches.
 - b. Find the area of a circle that has a radius of 4 inches.
 - c. Complete the statement: The area of a circle that has a radius of 4 inches is about _____ times as large as the area of a circle that has a radius of 2 inches.
8. What is the floor area of a circular tent that has a 14 foot diameter?
9. What is the area of a circular mirror whose diameter is $3\frac{1}{2}$ feet?
10. What is the area of the region enclosed by a semicircle and a diameter of a circle if the diameter is 56 inches?
- *11.
 - a. What is the area of the largest circle that can be drawn within a 14 foot square?
 - b. Is the area of the circle about $\frac{3}{4}$ of the area of the square?
- *12.
 - a. How does the area of a circle change if the radius of the circle is doubled?
 - b. How does the area of a circle change if the diameter of the circle is doubled?

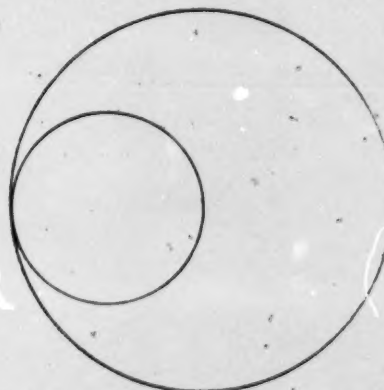
- *13. A rectangular plot of land, 40 feet by 30 feet, is mostly lawn. The circular flowerbed has a radius of 7 feet. What is the area of the portion of plot that is planted in grass?



- *14. The figure represents a simple closed curve composed of an arc of a circle and a diameter of the circle. The area of the region enclosed by this simple closed curve, measured in square inches, is 8π . Do not use any approximation for π in this problem. The symbol π will be a part of your answer.
- What is the area of the entire circle?
 - What is the second power of the radius of the circle?
 - How long is a radius of the circle?
 - How long is the straight portion of the closed curve represented in the figure?
 - What is the circumference of the (entire) circle?
 - How long is the arc represented in the figure?
 - What is the total length of the simple closed curve?



- *15. The center of the longer circle lies on the shorter circle. The intersection of the two circles is a single point. This point and the centers of the two circles lie on one line. If the interior of the shorter circle is chosen as a unit of measure, what would be the measure of the region inside the longer circle and outside the shorter circle?



14-7. Cylindrical Solids.

In Chapter 11 you studied the rectangular solid, its volume, and its surface area. In Chapter 13, you studied the prism, its volume, and its surface area. In this section you will study another solid which is used very often. Instead of having a rectangular region as a base, like a box, suppose a solid has a circular region as a base, like some tin cans. Such a solid is called a cylindrical solid (or sometimes just a cylinder). You have seen many other examples of cylindrical solids such as water pipes, tanks, silos, and some drinking glasses.

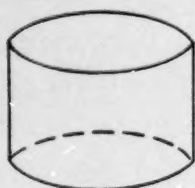


Figure 1



Figure 2

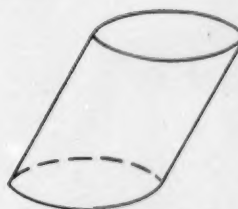


Figure 3

In the figures above, the first two drawings represent "right" cylinders. The third drawing represents a "slanted" cylinder. Slanted cylindrical solids are seldom used in ordinary life. The solids studied in this section will all be "right" cylindrical solids.

Look at Figures 1 and 2 carefully. Then try the exercises below.

Exercises 14-7a

Class Discussion.

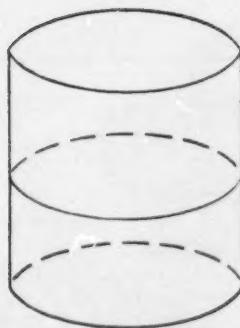
1. Complete the following statements.
 - a. Each cylinder has _____ bases.
 - b. The upper and lower bases represent regions of what shape?
 - c. The areas of these regions seem to be _____.
 - d. Each base is in a plane and the planes of the two bases are _____.
 - e. If the planes are regarded as horizontal, then the upper base is directly _____ the lower base.

- f. The lateral or side surface of the cylinder is made up of the points of segments. Each of these segments joins a point of the lower circle with the _____ directly above it on the upper circle.
2. Name some objects that you see in the classroom that suggest cylinders.
3. Name some objects that you have seen outside the classroom that suggest cylinders.
-

From Exercise 1 you learned some of the properties of right cylinders. A cylinder can be described by means of two lengths. These are the radius of the base of the cylinder and the altitude (or height) of the cylinder. The altitude can be thought of as the length of one of the segments lying in the lateral surface and joining the two bases of the cylinder.

How could you find the volume of a cylinder? Did you think of this fairly easy method? If the solid is like a tin can and will hold water (or sand) you can fill it up and then pour it into a standard container. Mathematicians like to know the answer without having to do this every time. For some cylinders this method would not be practical.

Remember how you found the volume of a box or of a (right) prism. You first considered a solid or prism one unit high. The number of cubic units in this solid is the same as the number of square units in the area of the base. You saw that the measure of the volume was the measure of the area of the base times one. Suppose the prism had an altitude of two units, then the measure of the volume would be twice as much as the measure of the area of the base. The measure of the volume would be 2 times the measure of the area of the base.



In general, if the area of the base is B square units and the altitude of the prism is h units, then the volume of the prism is $B \cdot h$ cubic units.

This method is also used to find the volume of a cylinder. The measure of the volume of the cylinder is the measure of the area of the base times the measure of the altitude. Since the base of a cylinder represents a circle, its area is πr^2 square units. Therefore the volume of a cylinder is $\pi r^2 \cdot h$ cubic units. Notice the kind of unit necessary to measure volume. It is a cubic unit.

There is one basic principle which applies to boxes, to other prisms, and to cylinders. The measure of the volume is the measure of the area of the base times the measure of the altitude.

To compute the volume of a cylinder, you multiply the measure of the area of the base by the measure of the altitude. This statement can be written in symbols as

$$V = \pi r^2 h$$

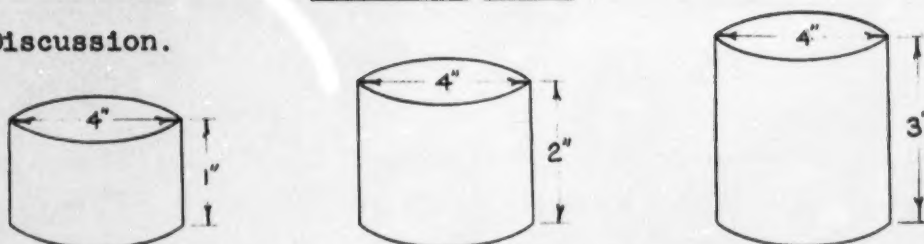
Remember: πr^2 represents the area of the base which is a circle, h represents the altitude (or height).

The radius and altitude are measured with the same kind of unit.

For most of you it is probably better not to memorize the formula as such. You should learn how to compute the volume of a cylinder. Think of the geometrical figure and what it is you want to find, then most problems of this type are easy.

Exercises 14-7b

Class Discussion.



Study the drawings above to learn how to find the volume of a cylinder.

1. What is the area of the base of cylinder A?
2. Are there as many cubic inches in cylinder A as there are square inches in its base? Why?
3. How many cubic inches are in cylinder A?
4. What is the volume of cylinder B?
5. What is the volume of cylinder C?
6. Complete the following statements.
 - a. The volume of cylinder B is _____ as large as the volume of cylinder A.
 - b. The volume of cylinder C is _____ times as large as the volume of cylinder A.
7. A cylinder is 8 feet high. The diameter of its base is 6 feet. How many cubic feet are in the cylinder?
8. Find the volume of a cylindrical can that is 5 feet high and has a diameter of 4 feet.

A note to save some work time. When you work problems using π , it may be easier to use a decimal approximation for π only at the last step of the arithmetic. In this way long decimals are used as little as possible.

Consider $\pi \cdot 5^2 \cdot 8$.

You know that $5^2 = 25$
 and that $25 \cdot 8 = 200$
 Therefore $\pi \cdot 5^2 \cdot 8 = \pi \cdot 200$
 $(3.14) \cdot (200) = 628$

Do you think that this method is easier than multiplying 3.14 by 25 and then multiplying the result by 8? Try it both ways and compare the amount of work.

Exercises 14-7c

1. A silo (with a flat top) is 30 feet high and the inside radius is 6 feet. How many cubic feet of green feed will it hold? (What is its volume?) Use $\pi \approx 3.14$.
2. The diameter of a tin can is 6 inches and its height 7 inches. What is its volume? Use $\pi \approx \frac{22}{7}$.
3. Information is given for four cylinders. The letters r , d , h , and V are the measures of the radius, the diameter, the altitude, and the volume respectively. Find the missing information. Leave your answers in terms of π . Example 15π .

r	d	h	V
	10	12	
	2	20	
$4\frac{1}{2}$		$5\frac{1}{2}$	
	$2\frac{1}{2}$	16	

4. Harry has two cans of juice. One can is 4 inches high and 3 inches in diameter, the other is 3 inches high and 4 inches in diameter. Which can holds more juice?
5. A cylindrical water tank is 8 feet high. The diameter of its base is 1 foot. Find the volume (in cubic feet) of water which it can hold.

6. There are about $7\frac{1}{2}$ gallons in a cubic foot of water. About how many gallons will the tank of the problem above hold?
7. Find the volume of a cylindrical solid whose altitude is 10 centimeters and the radius of whose base is 3 centimeters. Leave your answer in terms of π .
8. In Problem 7 above, what would the volume be if the altitude were doubled and the radius of the base were left unchanged? Find the ratio of the larger volume to the smaller.
9. In Problem 7 above, what would be the volume if the radius were doubled and the altitude were left unchanged? Find the ratio of the larger volume to the smaller.
10. In Problem 7 above, what would be the volume if the altitude were doubled and the radius of the base were also doubled? (Think of a new cylinder whose altitude is 20 centimeters and the radius of whose base is 6 centimeters.) Find the ratio of the larger volume to the smaller.
- *11. Use your results from Problems 7, 8, 9, and 10 to help you complete the following statements.
 - a. When the altitude is doubled the volume is _____.
 - b. When the radius of the base is doubled the volume is _____.
 - c. When both the altitude and the radius of the base are doubled the volume is _____.
- *12. A cylindrical tank has a 5 foot diameter. It is filled with water to a height of 20 inches. What is the volume of the water in cubic inches?
- *13. Find the amount of water (volume in cubic inches) which a 100 foot length of pipe will hold if the inside radius of a cross section is 1 inch. Use $\pi \approx 3.14$. (A cross section is shaped like the base. A cross section is the intersection of the solid and of a plane parallel to the planes of the bases between them.)

14-8. Cylindrical Solids--Surface Area.

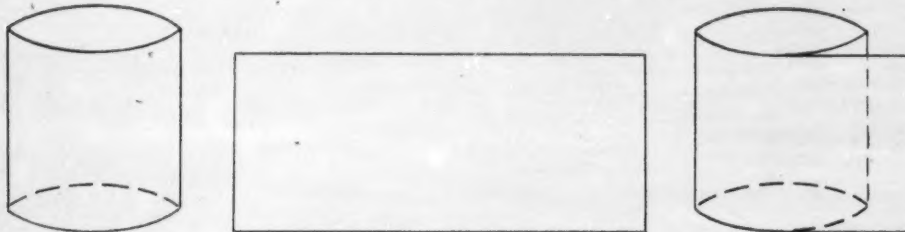
In Section 14-7 you learned about the volume of a cylinder. Now you will learn about surface area. There are two questions you might ask.

1. What is the surface area of the curved part (the lateral area)?
2. What is the total surface area?

Now think of the cylinder. The total area is the lateral area plus the area of the top base plus the area of the bottom base. But the areas of the top and bottom bases are the same. The measure of the area of each base is πr^2 where r is the measure of the radius of the base. If you know how to find the lateral surface area, you also know how to find the total area. The following experiment should help you find the lateral area of a cylinder.

Exercises 14-8a

1. Experiment 3. For this experiment you will need a tin can and its paper label. The label on the can covers the lateral surface of the cylinder.
 - a. Cut the label along its full height. Lay the label out flat. (as shown in the diagrams below.)



Complete the following statements.

- b. The label is made and printed in the form of a _____.
- c. The height of the rectangle is the _____ of the cylinder.
- d. The length of the rectangle is the _____ of the base circle of the cylinder.

- e. The length of the rectangle is the same as _____ times the diameter of the cylinder.
 - f. The area of the rectangle, using the symbols π , d , and h , is _____.
 - g. The area of the rectangle, using the symbols π , $2 \cdot r$, and h is _____.
2. Is this a true statement? "The lateral area of a cylinder is the area of a rectangle which will just cover it."

Did you notice these facts as a result of your experiment?

1. The lateral area of a cylinder is the area of a certain rectangle.
2. The altitude of the rectangle and the altitude of cylinder are the same.
3. The length of the base of the rectangle and the length of the base circle of the cylinder are the same.

Now you know that the measure of the lateral surface area of the cylinder is the product of the measure of the circumference of the base circle and the measure of the height. We could write that the measure is

$$2\pi r \cdot h \quad \text{or} \quad \pi \cdot d \cdot h.$$

Tell why these are the same.

Exercises 14-8b

Class Discussion. Use $\pi \approx 3.14$.

1. What is the lateral area of a cylindrical water tank that is 20 feet in diameter and 20 feet high?
2. Find the lateral surface area of a cylinder whose altitude is 8 centimeters and the radius of whose base circle is $1\frac{1}{2}$ centimeters.

Earlier in this section we decided that we could find the

total area if we knew how to find the lateral surface area. The total area is the lateral area plus the area of the top base plus the area of the bottom base. These bases represent circles and have the same area.

3. Complete the following statements.
- The area of one base is _____.
 - The area of two bases is _____.
 - The lateral area is _____.
 - The total area is _____ + _____.

All curved surfaces cannot be treated in such a simple way. Think of the surface of a ball. Rectangular regions, or other flat surfaces, just don't wrap nicely around balls. The areas of such surfaces can be found in other ways but we will not learn about them this year.

Exercises 14-8c

- Find the total surface area of a cylinder whose altitude is 7 inches and the radius of whose base is 5 inches.
Use $\pi \approx 3.14$.
- Find the total area of the cylinder in Problem 1, Section 10-8b.
- Find the total area of the cylinder in Problem 2, Section 10-8b.
- *4. A small town has a large cylindrical water tank that needs painting. A gallon of paint covers about 400 square feet. How much paint is needed to cover the whole tank if the radius of the base is 8 feet and the height of the tank is 20 feet? Give your answer to the nearest tenth of a gallon.

14-9. Summary.

The circle is a simple closed curve in a plane. It may be described as a set of points. Each point of the circle is the same distance from a certain point called the center. The center is not in the set of points that make up the circle, but it is in the same plane.

The compass may be used to draw circles and to transfer distances.

A radius of a circle is one of the segments joining a point of the circle and the center.

A diameter of a circle is a line segment which contains the center of the circle and whose end-points are on the circle.

The words radius and diameter are used to mean either the segment or the length of the segment described above.

A circle has many radii and many diameters.

If r is the measure of the radius of a circle and d is the measure of the diameter of the same circle, then $d = 2 \cdot r$ or $r = \frac{1}{2} \cdot d$.

If the intersection of a line and a circle is a single point, then the line is tangent to the circle. The single point of their intersection is called the point of tangency.

An arc is a portion of a circle. Two different points are necessary to separate a circle into arcs.

An arc is a semicircle if the end-points of the arc are also end-points of a diameter of the circle.

An angle whose vertex is at the center of a circle is called a central angle.

The length of a circle is called its circumference.

The measure of the length of a circle is $2\pi r$ or πd .

3.14 or $\frac{22}{7}$ can be used as approximations for the number π .

The area of a circle is π times the area of a square whose side is the length of the radius of the circle. This statement can be written in symbols as $A = \pi r^2$.

You studied several formulas in this chapter. You should know these two without fail.

1. The measure of the length of a circle is $2\pi r$ (or πd).
2. The measure of the area of a circular region is πr^2 .

The formulas for the measures of the volume and surface area of a cylindrical solid need not be learned, as such, if you are able to think of what the formulas represent.

You should understand and know the basic general principle for getting the volume of many solids. For boxes, for other prisms, and for cylinders this principle tells you that the measure of the volume is the measure of the area of the base times the measure of the altitude in the same unit.

In many problems such as surface area you should think of the geometric objects and what it is about these that you want to know. For example, if you are interested in the lateral area of a cylinder, you should think of what the lateral surface represents when it is flattened out. Then you would get the lateral area as the measure of the length of the base circle times the measure of the height of the solid. For the measure of the total surface area you need only add to this measure twice the measure of the area of the base.

The circle is the simplest of all curves and the one most frequently used. You have studied here only a few of the most elementary facts about it. It is interesting to realize that to calculate its circumference and area requires the use of a number whose study has taxed the abilities of mathematicians for centuries.

14-10. Chapter Review.Exercises 14-10

1. Complete the following statements.
 - a. A circle is a _____ curve.
 - b. Any line segment through the center of a circle and having its end-points on the circle is a _____ of the circle.
 - c. The _____ of a circle is twice as long as the _____ of the same circle.
 - d. Any part of a circle is called a(n) _____.
 - e. The length of a circle is called its _____.
 - f. A circle whose diameter is 5 inches has a radius that is _____ inches.
 - g. A line that intersects a circle in exactly one point is called a _____.
2. Draw three concentric circles in which the diameter of the smallest circle is 2 inches.
3. Find the missing information. Use $\pi \approx 3.1$.

r	d	c	A
14			
	14		
		51	
17.5			
10			314

4. What is the circumference of a circle with a radius of $24\frac{1}{2}$ inches?
5. A circular flower garden has a radius of 12 feet. What is its diameter?
6. A wheel on Harry's wagon has a diameter of 15 inches. What is the circumference of the wheel?

7. A circular fish pond has a diameter of $9\frac{1}{2}$ feet. How far is it from the center of the fish pond to the edge of the pond?
8. What radius should be used to draw a circle that will have a circumference of 22 inches?
9.
 - a. What is the area of a circle whose radius is 20 feet?
 - b. What is the area of a circle whose radius is 10 feet?
 - c. Compare the areas of the two circles.
10. A circular table top is 27 inches in diameter. What is the area of the table top?
11. What is the area of a mirror that is a circle 21 inches in diameter?
12. The center-jump circle on a basketball court has an inside diameter of 4 feet. What is the area of the circle?
13. Find the volume of a cylindrical can that is 5 inches high and has a diameter of 4 inches.
- *14. Round concrete pillars 14 inches in diameter and 18 feet high are used as supports for a boat landing. How many cubic feet of concrete will there be in one of these pillars?

14-11. Cumulative Review.

Exercises 14-11

How Are You Doing?

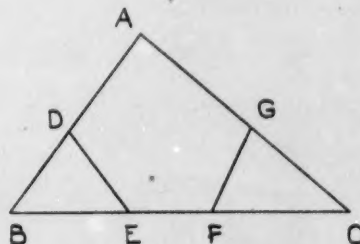
1. Which of the following statements is not true.
 - a. The difference between any two prime numbers greater than 5 is always an even number.
 - b. A multiple of a prime number is also a prime number.
 - c. Zero is a multiple of all counting numbers.
2. Write the set of factors of 77.
3. Express 26 as the sum of 2 prime numbers.

4. What is the least common multiple of 3, 6, and 12?
5. The value of 1^{30} is ____.
6. Round each of the following as indicated.
 - a. 0.04753 (nearest thousandth)
 - b. 628.3849 (nearest hundredth)
7. Perform the following operations.
 - a. 16.532×100
 - b. $349.1 \div 1000$
8. Find the rate of interest if the interest on \$2500 for one year was \$112.50.
9. What is the cost of 15 candy bars at 3 for 25¢?
10. Multiply, then express the result in the best form.
 - a. $\frac{3 \text{ T. } 500 \text{ lbs.}}{2}$
 - b. $\frac{3 \text{ qt. } 1 \text{ pt.}}{3}$

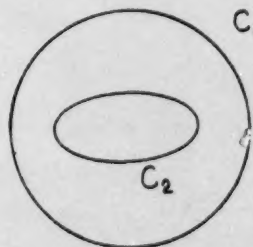
11. In the figure at the right,

what is

- a. $\overrightarrow{BA} \cup \overrightarrow{BF}$?
- b. $\overrightarrow{BF} \cup \overrightarrow{EC}$?
- c. $\overrightarrow{ED} \cap \overrightarrow{AB}$?
- d. $\overrightarrow{AG} \cap \overrightarrow{DE}$?
- e. $\overrightarrow{DE} \cap \overrightarrow{BC}$?
- f. $\overrightarrow{DE} \cap \triangle ABC$?



12. In the figure at the right the two simple closed curves are represented as C_1 and C_2 .



- a. Make a copy of this figure.
- b. Shade the exterior of C_2 with horizontal lines.
- c. Shade the interior of C_1 with vertical lines.
- d. Using the word "intersection" describe in words the region that is doubly shaded.

13. Complete:

- a. 7.2 M. = _____ cm.
- b. 47 mm. = _____ cm.
- c. 246 in. = _____ ft.
- d. $4\frac{1}{4}$ mi. = _____ ft.

14.

- a - _____
- b _____
- c _____

Which of the above segments is nearest the length of one centimeter?

15. How many pairs of angles in the drawing are

- a. adjacent angles
- b. supplementary angles
- c. corresponding angles
- d. vertical angles



16. The distance from the center of a wheel to its circumference is 33 inches. What is the length of the wheel?

Chapter 15

STATISTICS AND GRAPHS

15-1. Introduction.

In this chapter we will talk about statistics. The word has two different meanings. It is sometimes used to describe a method of studying facts that can be stated in numbers. Statistics is also used to mean numerical facts. Persons who work in this field are known as statisticians. The statistician does the following.

1. Decides the kinds of information needed.
2. Directs the collection of the information.
3. Organizes or classifies the information in a suitable form (like a graph or a table).
4. Interprets results. (Draws conclusions.)

It is important to understand that none of the methods of statistics is perfect. There may be some mistakes, faulty reasoning or incorrect conclusions. The beginning facts must be accurate, the methods properly applied and the results explained.

We will use some methods of the statistician as we go through this chapter.

15-2. Gathering Data.

Look at the other pupils in your classroom. Notice that some pupils are about the same height. Notice also that some pupils are taller than others, and some are much shorter.

What is the height of the tallest pupil in your room?

What is the height of the shortest pupil?

How many pupils are the same height?

To answer these questions you would have to measure the height of each pupil. When you have measured all of the pupils, you have gathered facts about their height. The facts that you have gathered are called data (dā' tā).

Suppose you measure to the nearest inch the heights of 15

boys in your school room. You might have a group of numbers such as this: 55, 61, 59, 52, 61, 54, 61, 63, 53, 64, 57, 54, 55, 65, 60. It would be difficult to work with these numbers unless they are arranged in some way.

The information might be rearranged and put in order of size: 65, 64, 63, 61, 61, 61, 60, 59, 57, 55, 55, 54, 54, 53, 52. This order will help you to get information from the measurements more easily. This same information is even easier to see and use in the form of a table. The pupils could be listed by name, however, in this table each boy is given a number.

Heights of 15 Seventh Grade Boys	
Boy	Heights in Inches
1	65
2	64
3	63
4	61
5	61
6	61
7	60
8	59
9	57
10	55
11	55
12	54
13	54
14	53
15	52

Table 15-2a

Now it is easier to answer questions about this data.

Exercises 15-2a

Class Discussion.

1. What is the height of the tallest boy?
2. What is the height of the shortest boy?
3. How many boys are 61 inches tall?
4. What do you notice about the number of boys that are 55 inches tall and the number of boys that are 54 inches tall?

5. How many boys are more than 60 inches tall?
6. How many boys are less than 60 inches tall?

Gathering data about the heights of the pupils in a class is an example of one kind of thing that is done when you study statistics. Another thing is arranging the tables showing the data. Tables and charts make it easier for the reader to understand information that has been gathered.

Suppose a junior high school was opened as a new school in 1955 with 500 students. In 1956 the enrollment was 625; in 1957 it was 1000; in 1958 it rose to 1125; by 1959 the school had 1250 students; and when school began in 1960, the enrollment was 1500. Making a table of this information makes it easier to read.

Junior High School Enrollment

Year	Number of Students
1955	500
1956	625
1957	1000
1958	1125
1959	1250
1960	1500

Table 15-2b

The general trend is clear. The school enrollment is growing rapidly.

Often it is not possible to get information every year. There could be intervals of 2 years, 5 years, 10 years, or even 50 or 100 years, depending on the type of information. For example, the United States census is taken every 10 years. The following table shows an example of 10-year intervals.

Growth of Scranton, Pennsylvania*

Year	Population
1880	45,850
1890	75,215
1900	102,026
1910	129,867
1920	137,783
1930	143,433
1940	140,404
1950	125,536

*(1960 World Almanac, page 302)

Table 15-2c

Exercises 15-2b

Class Discussion.

Use the table above to help you answer these questions.

1. What was the largest population recorded? In what year?
2. What seemed to be happening to the population until 1930?
3. What seemed to be happening to the population after 1930?
4. Do you think that this change took place exactly in the year 1930?
5. Do you know enough to guess about 1929? about 1930?

Sometimes information is arranged showing how a total is divided into parts. For example the information below shows how the total passenger car production for 1959 was divided among various companies.

Table 15-2d

Passenger Car Production, U. S. Plants, 1959*

Company	Number of Cars
American Motors Corp.	217,332
Chrysler Corp.	581,300
Ford Motor Co.	1,219,422
General Motors Co.	2,169,186
Studebaker-Packard Corp.	56,920
Checker Cab	3,267
Grand Total	4,247,427

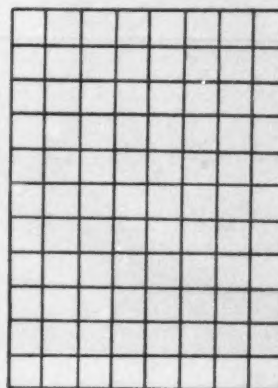
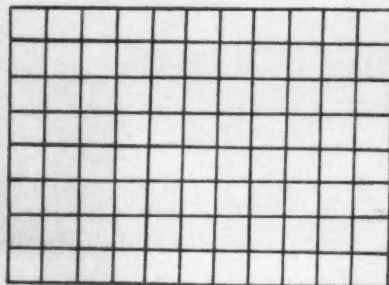
*(1960 World Almanac, page 698) 33

15-3. Bar Graphs.

Such data as was discussed in the above tables are often seen by "drawing a picture" of the information. In Table 15-2a the heights of 15 boys are listed. One way to show this in "picture" form would be to have the boys line up against the wall in order of height, and show a picture of this. An easier way is to represent each boy by a "bar." Each bar should be labeled boy 1, boy 2, up to boy 15. The heights should be marked along the left side of the paper. Since you want to tell an accurate story of the information, it is wise to use paper which has horizontal and vertical lines. This is called graph paper. You must mark off these units carefully for you do not want the bars to be so small that they are lost on the page. On the other hand, you do not want to have your graph almost completed and then discover that one bar is too long for your paper.

This type of graph, known as the bar graph, usually compares objects. The bars may be either horizontal or vertical. This is usually decided by the size of the numbers in the information you have.

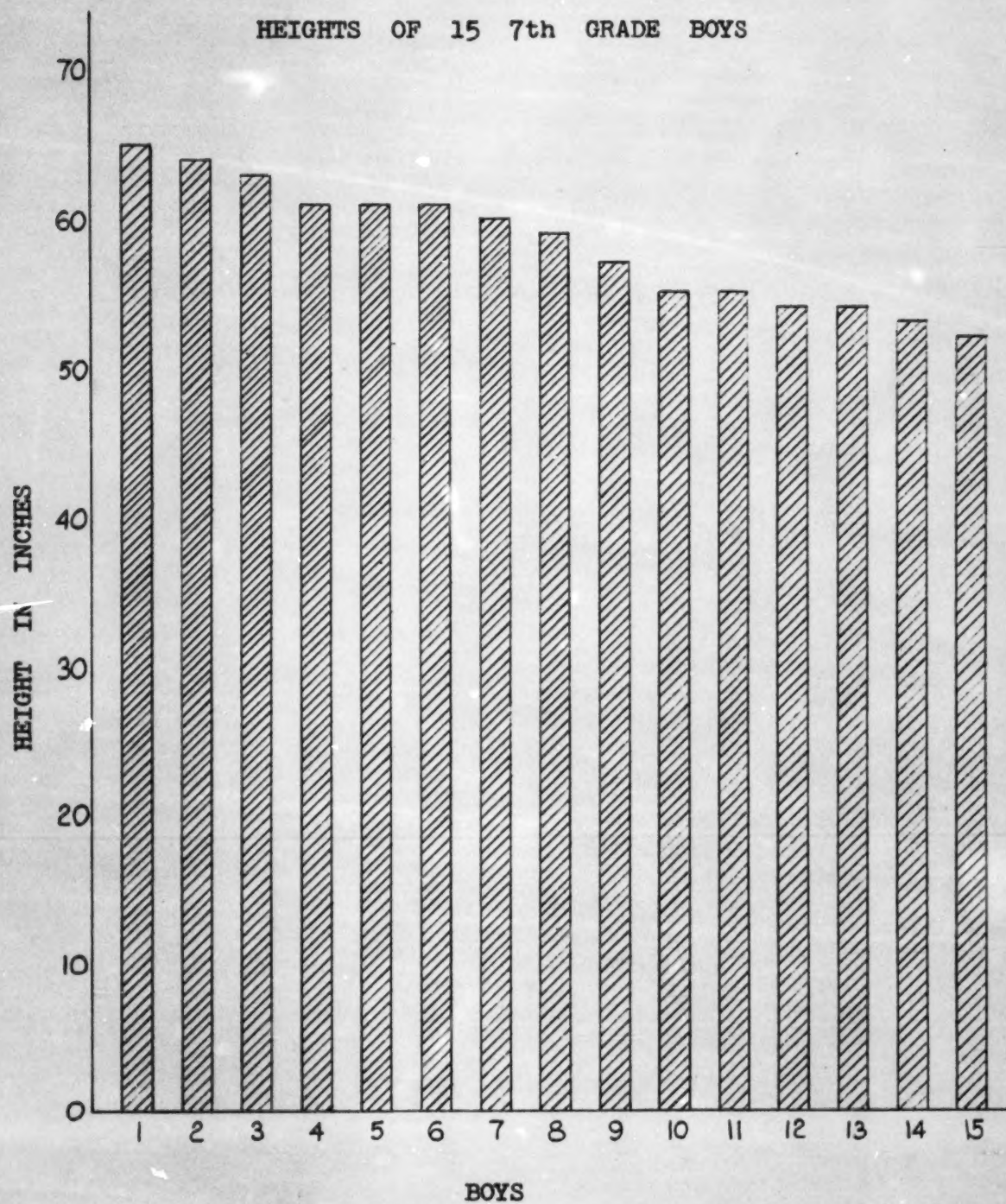
Your paper may be turned either way. Let the size of your numbers determine this, also.



Look carefully at the following graph and notice that:

1. Room is allowed for all labeling.
2. The graph tells a complete story.
3. Marking the title and the units is absolutely necessary.
4. The units are equally spaced.
5. The base line is chosen near the bottom of the page.
6. The proper units are written beneath the base line and to the left of the vertical line.
7. Room is allowed at the top for the title.

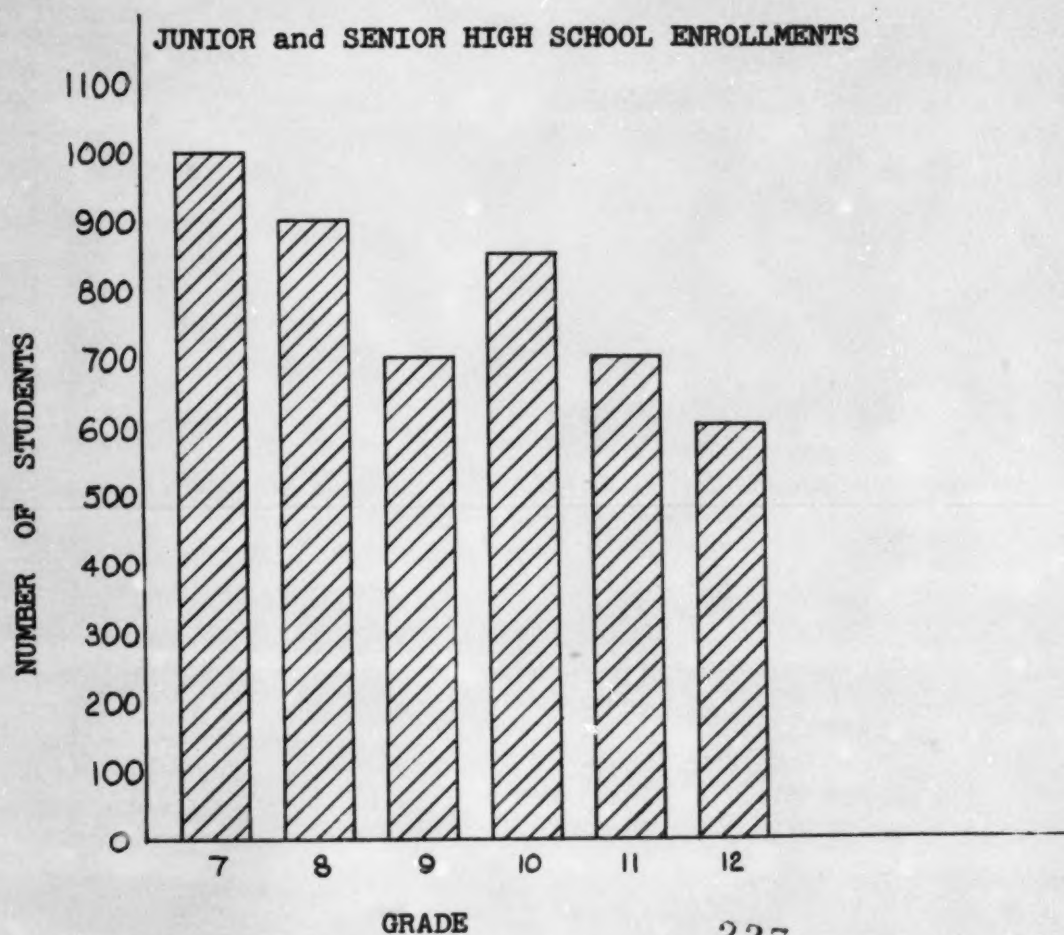
Figure 15-3



With the bar graph it is easier to compare the heights of the boys than it was when we had only the table. Think of each bar as representing a boy lined up against a wall.

Exercises 15-3

1. Suppose the heights of seven students in your classroom were (in inches) 64, 57, 53, 68, 55, 57, 59.
 - a. Arrange these numbers in a table (similar to Table 15-2a).
 - b. Make a bar graph to show this information.
2. Study the following graph and then answer the questions below the graph.



- a. Which grade has the largest number of students?
 - b. How many students are in the above grade?
 - c. Which grade has the smallest number of students?
 - d. How many students are in this grade?
 - e. Do any grades have about the same number of students?
If so, which ones do?
3. During a showing of a movie a theater had 450 men, 700 women, and 350 children see the picture. Draw a bar graph based on this information.
 4. Find out the enrollment (number of students) of each grade in your school. Draw a bar graph based on this information.
 5. Find out how many students in your school are absent each day for a week. Make a bar graph of this information.
 6. Find out how many students in your school are tardy each day for a week. Make a bar graph of this information.
 - *7. Draw a bar graph to represent the number of people killed in different types of accidents during 1956 as shown in this table.

Motor vehicle accidents	40,000
Falls	20,200
Fires and injuries from fires	6,500
Drownings	6,100
Railroad accidents	2,650

15-4. Broken Line Graphs.

The broken line graph frequently shows progress or growth. It also is a "picture" of the information. With this "picture" the information can be more easily understood. Again, as in the bar graph, you will have to choose your units carefully so your picture will show exactly what you want it to show. Information is put on the graph by marking points. These points are joined by lines.

How would you draw a "picture" of the information in Table 15-2b? First choose your units, mark them, and place your points on your graph.

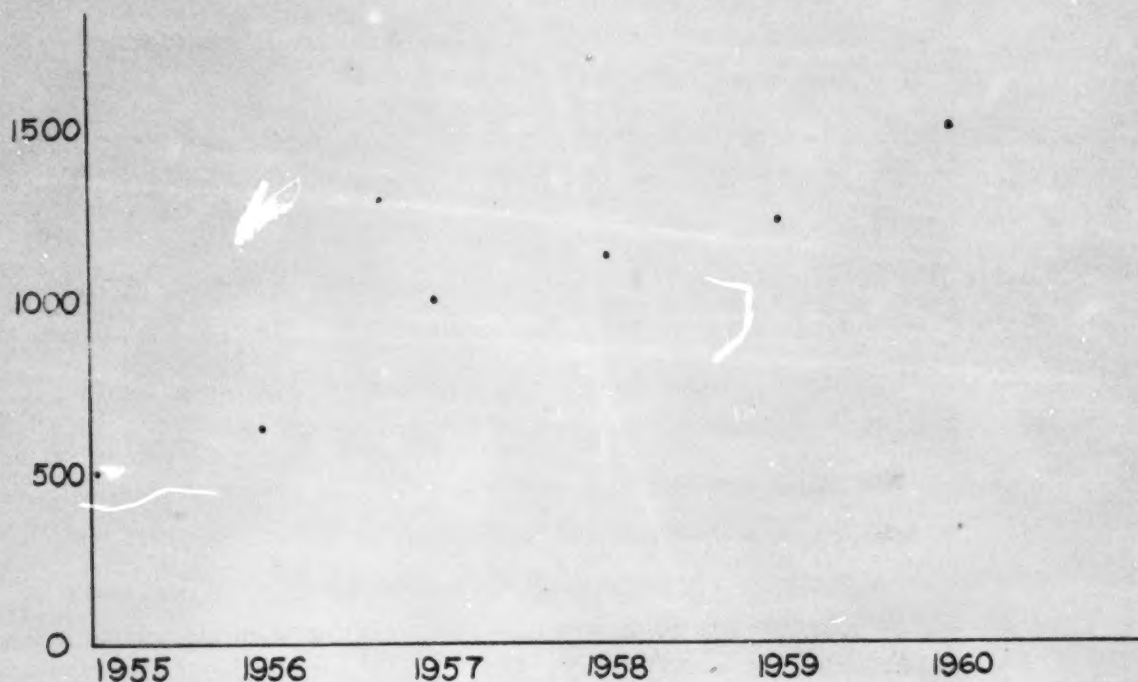


Figure 15-4a

Next, join these points by drawing line segments. Now label your units more completely and write your title neatly. With this broken-line graph, you can readily see not only that the school is growing quite rapidly, but also during what periods the growth was the greatest.

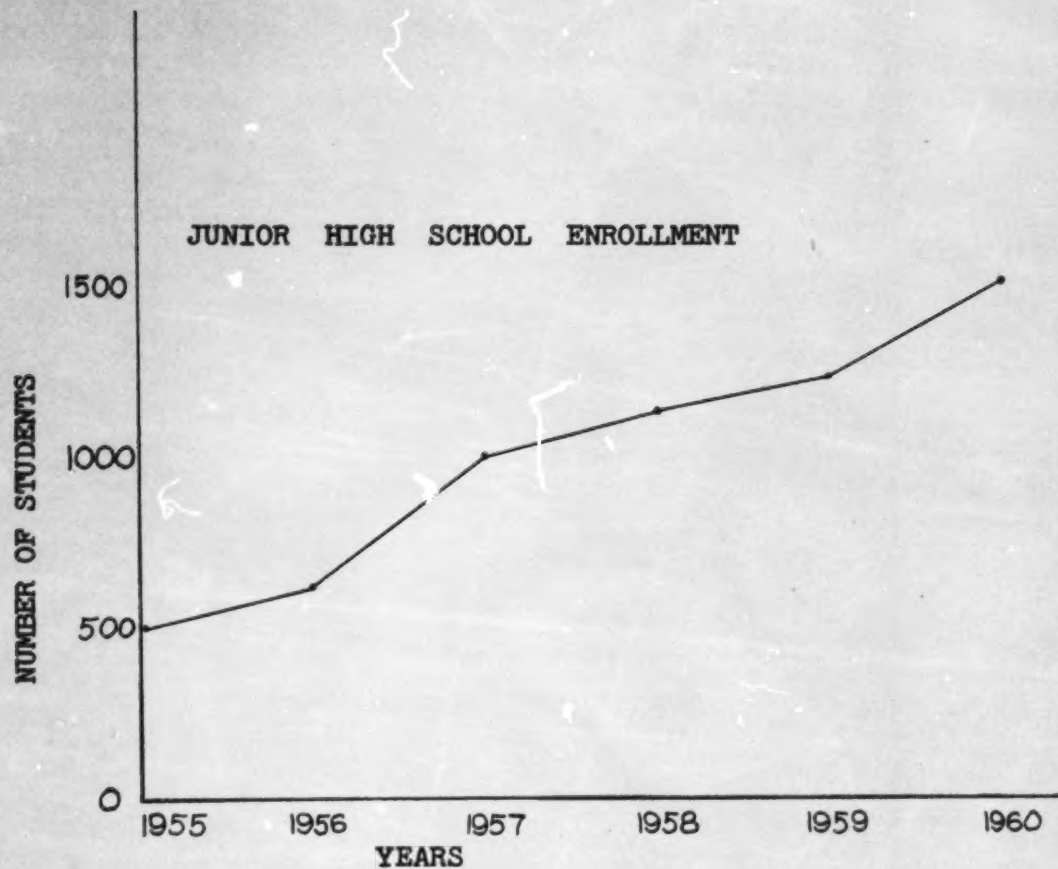


Figure 15-4b

Does this broken-line really represent the exact number of students between any two years? No, the points in Figure 15-4a are connected because the "picture" in Figure 15-4b shows more clearly how the school enrollment is changing. The information is not increased by drawing the lines.

In Table 15-2c, you saw a table expressing a population growth by 10-year periods. A broken line graph of this table might look like the one below.

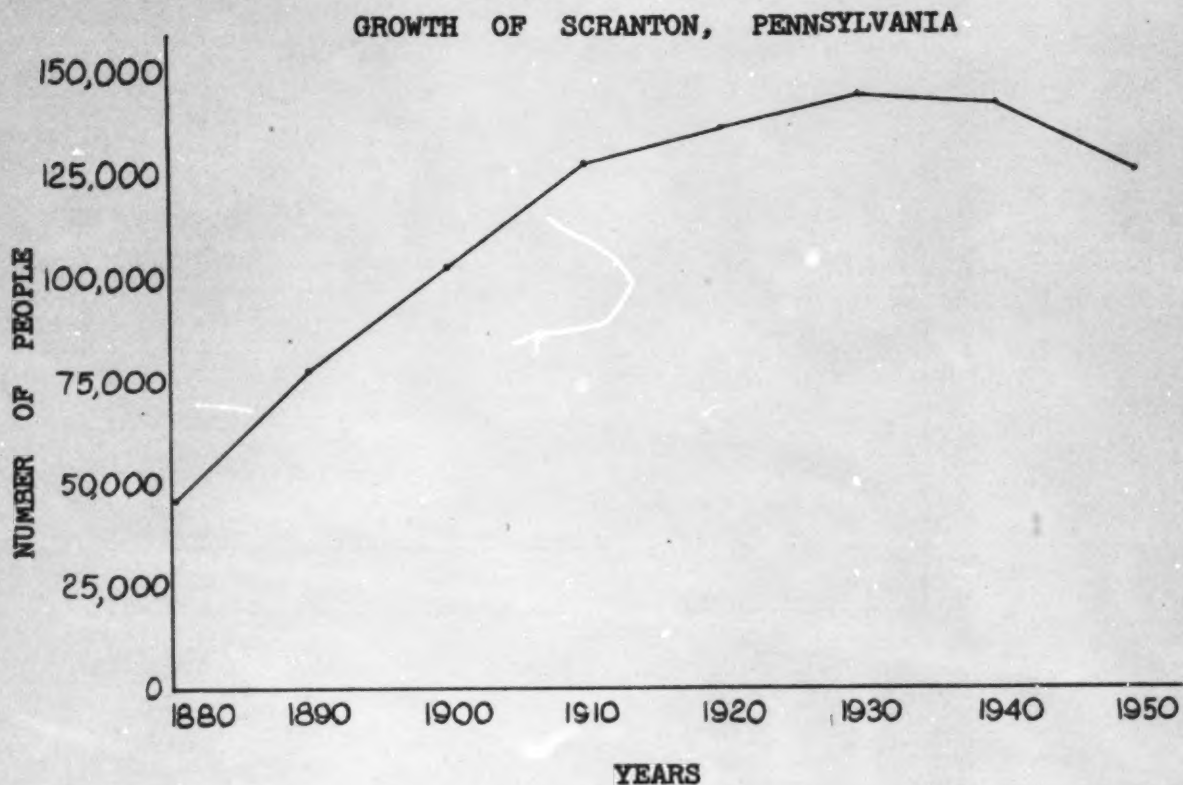


Figure 15-4c

When you deal with numbers this large, it is often difficult to decide upon the proper unit. Writing out the large numbers at the side looks clumsy. Many times these numbers are written in a different way. Look at the following table with very large numbers.

Table 15-4d
Growth of New York City*

<u>Census Years</u>	<u>Population</u>
1880	1,911,698
1890	2,507,414
1900	3,437,202
1910	4,766,883
1920	5,620,048
1930	6,930,446
1940	7,454,995
1950	7,891,957

You can express this information concerning the population by rounding the population figures to thousands of people. Then the table looks like the one below.

Table 15-4e
Growth of New York City

<u>Census Years</u>	<u>Population in Thousands</u>
1880	1,912
1890	2,507
1900	3,437
1910	4,767
1920	5,620
1930	6,930
1940	7,455
1950	7,892

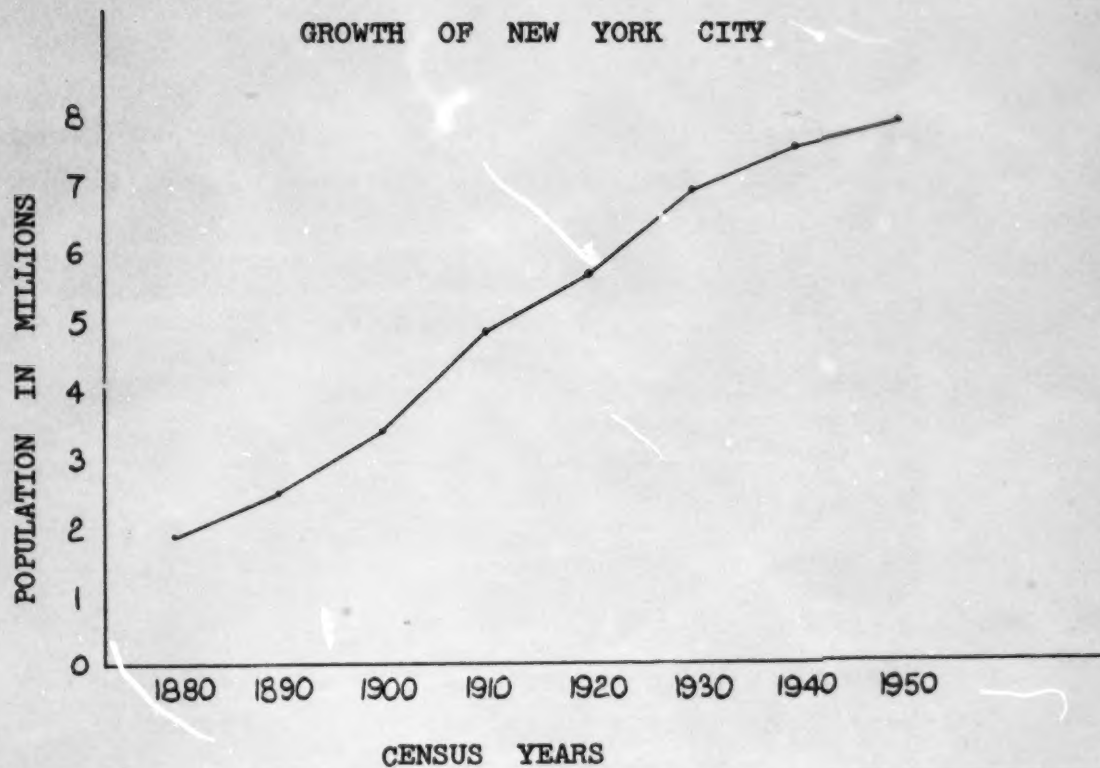
Even these units seem rather large for a graph. Look back at Table 15-4d. Now express the population in terms of the nearest million and the table will look like this:

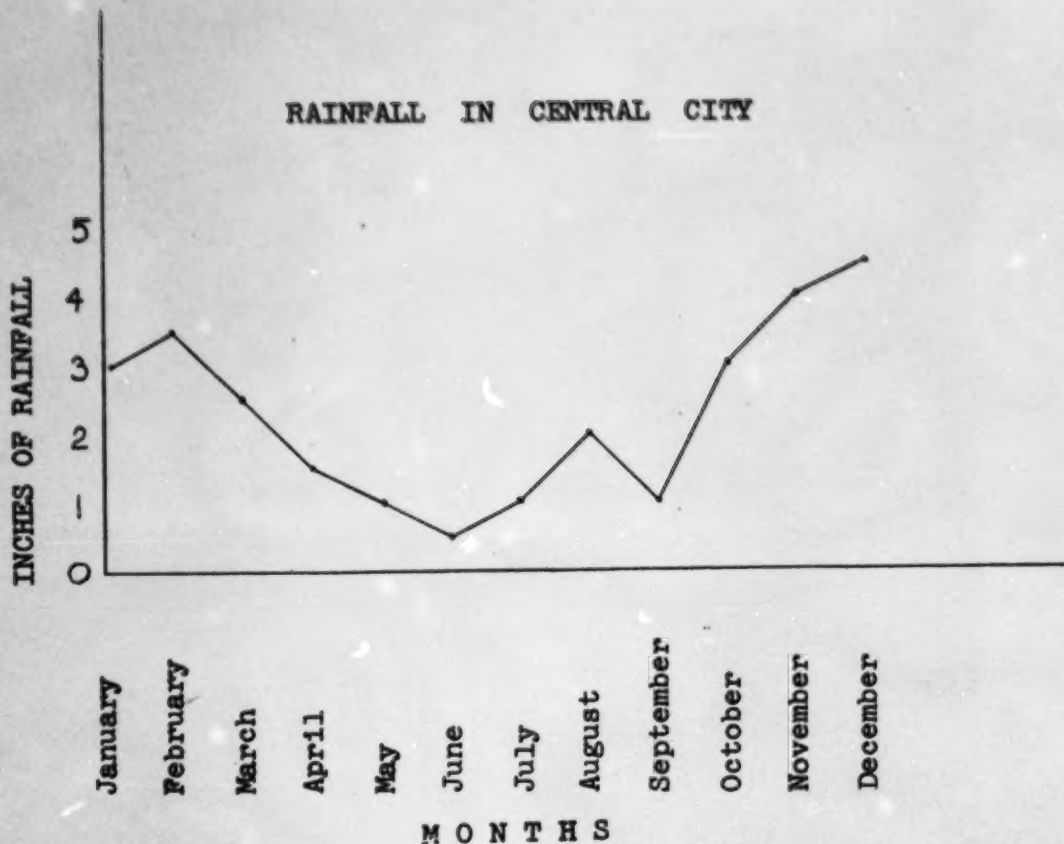
Table 15-4f
Growth of New York City

<u>Census Years</u>	<u>Population in Millions</u>
1880	1.9
1890	2.5
1900	3.4
1910	4.8
1920	5.6
1930	6.9
1940	7.5
1950	7.9

*(1960 World Almanac)

Making a broken line graph of this now becomes much simpler, for the numbers are small. A word of warning! Label these new units correctly.



Exercises 15-4

1. Use the above graph for this exercise.
 - a. What month had the greatest amount of rainfall?
 - b. What month had the smallest amount of rainfall?
 - c. Complete the table below. Read the graph as accurately as you can.

Rainfall in Central City, 1959

Months	Inches of Rainfall
January	
February	
March	
April	
May	
June	
July	
August	
September	
October	
November	
December	

2. Draw a broken-line graph from the following table.

Growth of Schools in Union County

Year	Number of Schools
1920	4
1930	6
1940	10
1950	17
1960	25

3. Make a broken line graph to represent the data given in this table.

The number of students in Franklin
Junior High School in years 1952-1957.

1952-- 86	1955-- 196
1953-- 150	1956-- 235
1954-- 164	1957-- 254

- *4. There are 83 National Monuments* in the United States. The ten largest in total number of acres are as follows in alphabetical order: Badlands, S.D., 98,646 acres; Canyon de Chelly, Ariz., 83,840 acres; Death Valley, Calif.-Nevada, 1,879,048 acres; Dinosaur, Utah-Colorado, 190,962 acres; Glacier Bay, Alaska, 2,274,248 acres; Grand Canyon, Arizona,

*(1960 World Almanac, p.219)

193,040 acres; Joshua Tree, California, 505,132 acres; Katmai, Alaska, 2,697,590 acres; Organ Pipe Cactus, Arizona, 328,691 acres; and White Sands, New Mexico, 140,247 acres. Arrange them in a table according to the number of acres. Start with the largest, and end with the smallest as the tenth on the list. Write another column in the table where the acreage is expressed in thousands of acres.

- *5. Fill in the following table.

Growth of Los Angeles*

<u>Census Yrs.</u>	<u>Population</u>	<u>Pop. in thousands</u>	<u>Pop. in ten thous.</u>	<u>Pop. in h. thous.</u>
1880	11,183			
1890	50,395			
1900	102,479			
1910	319,198			
1920	576,673			
1930	1,238,048			
1940	1,504,277			
1950	1,970,358			

6. Make a broken line graph to represent the data in this table.

Popular Vote in Millions Cast for Presidential Candidates of the U.S. 1928 to 1956

<u>Year</u>	<u>Republican Party</u>	<u>Democratic Party</u>
1928	21.4	15.0
1932	15.8	22.8
1936	16.7	27.5
1940	22.3	26.8
1944	22.0	24.8
1948	22.0	24.1
1952	33.8	27.3
1956	35.6	25.7

Observe these instructions.

In the same graph draw one broken line (in blue) for the Republican party and one (in red) for the Democratic party.

*(1960 World Almanac p. 302)

15-5. Circle Graphs.

A circle graph is shown in Figure 15-5a. Such a graph is used to show the comparison between the parts of a total and between the total and any part. This graph shows the percents of income which a family spent for food, clothing, rent and miscellaneous expenses and the percent of income which was put into savings.

HOW ONE FAMILY SPENDS ITS MONEY

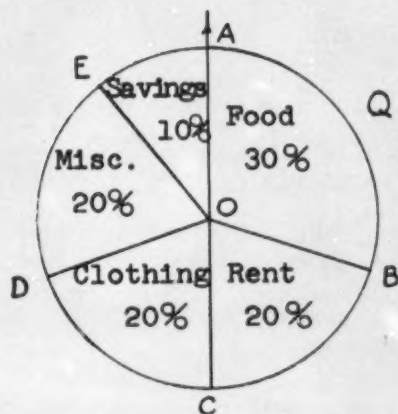


Figure 15-5a

The family spent 30 percent of its income for food, 20 percent for clothing, 20 percent for rent, 20 percent for miscellaneous expenses and saved 10 percent. The family's total income is represented by the area of the circle. Any length can be chosen for the length of the radius. The length is usually chosen so that the circle is large enough to show clearly the parts into which it is divided, and small enough to fit into the space.

Since the part of the income spent for food is 30 percent we must have an area which is 30 percent of the circle's area to represent this part of the income. To get this area we may begin by drawing the ray $\overline{OP}$ (See figure 15-5a) in vertical position. Label with A the intersection point of the ray and the circle. We know that we can divide the circle into 360 equal parts by drawing 360 angles of 1 degree, each with its vertex at O. Thirty percent of 360° is 108° . We draw a central angle of

108° with one side on OA. The other side is OB. The angle AOB is 30 percent of the area of the circle. Hence the interior of the closed curve OAQB represents the part of the income spent for food. In the same way, the sizes of the other parts represent the part of the income that is spent for rent, clothing, savings and miscellaneous.

How many degrees are there in each of the following: $\angle BOC$, $\angle COD$, $\angle DOE$, $\angle EOA$? Use a protractor to check your answer to this question.

If the family's income is \$6,000, how much is spent for food? How much is put into savings?

In a certain school there are 480 pupils. At lunch time 80 pupils go home for lunch, 120 bring their lunch, and 280 buy their lunch in the school lunchroom. The circle graph in Figure 15-5c shows the way the pupils are divided at lunch time. Before making this circle graph we had to find either what percent or what fractional part of the pupils go home for lunch, bring their lunch, and buy their lunch. These fractions and percents are shown in the table.

Table 15-5b

<u>Number of Pupils</u>		<u>Fractional Part of Total</u>	<u>Percent</u>	<u>Degrees</u>
Buy lunch	280	$\frac{280}{480} = \frac{7}{12}$	$58\frac{1}{3}$	210
Bring lunch	120	$\frac{120}{480} = \frac{1}{4}$	25	90
Go home	80	$\frac{80}{480} = \frac{1}{6}$	$16\frac{2}{3}$	60
Totals	480	1	100	360

SOURCES OF PUPIL'S LUNCHES

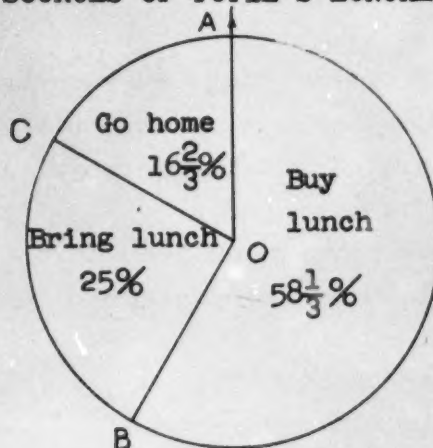


Figure 15-5c

This is how the sizes of the angles were found.

$$\angle AOB : \frac{7}{12} \text{ of } 360^\circ = 210^\circ$$

$$\angle BOC : \frac{1}{4} \text{ of } 360^\circ = 90^\circ$$

$$\angle COA : \frac{1}{6} \text{ of } 360^\circ = 60^\circ$$

Ray $\overline{OA}$ was drawn in vertical position. The largest angle, $\angle AOB$, was measured in a clockwise direction. Continuing measuring in this direction the next largest, $\angle BOC$, was measured and then the smallest, $\angle COA$. The names of the parts and the corresponding percentages were then indicated on the graph. Note that while we need the number of degrees in order to draw the angles, we label only the percentage. The percentages mean more to the reader than the degrees.

Table 15-5d (Passenger Car Production) is not an easy one to make into a circle graph because of the big numbers and the large amount of computation. However, it brings up an interesting point. Some numbers are so small compared with the large ones that they cannot show up on the graph at all.

Table 15-5d
 Passenger Car Production, U.S. Plants, 1959

<u>Company</u>	<u>Number of Cars</u>	<u>Fractional Part of Total</u>	<u>Nearest Percent</u>	<u>Number of degrees</u>
General Motors Corp.	2,169,186	$\frac{2,169,186}{4,247,427}$	51	184
Ford Motor Co.	1,219,422	$\frac{1,219,422}{4,247,427}$	29	104
Chrysler Corp.	581,300	$\frac{581,300}{4,247,427}$	14	50
Amer. Motors Corp.	217,332	$\frac{217,332}{4,247,427}$	5	18
Stude.-Packard Corp.	56,920	$\frac{56,920}{4,247,427}$	1	4
Checker Cab	3,267	$\frac{3,267}{4,247,427}$	(less than 1)	
Totals	4,247,427	1	100	360

Passenger Car Production
 U.S. Plants, 1959

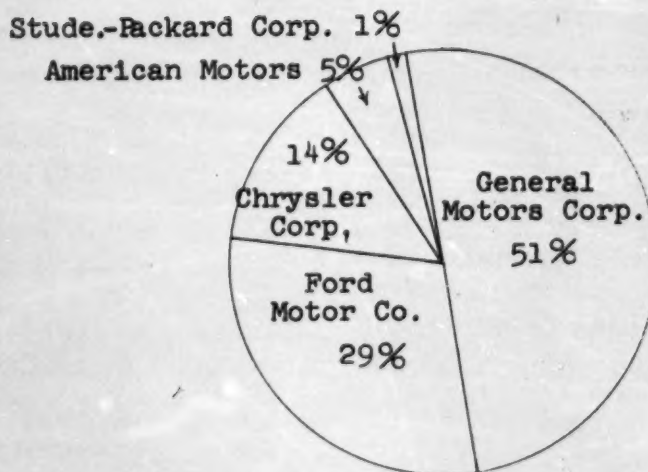


Figure 15-5e

Exercises 15-5

Make a circle graph to represent the information in each of the following problems. Make one graph for each problem after you have completed the table of information.

1. In 1949 it was found that school-related accidents which involved seventh grade pupils in the U.S. happened as shown. (Figures given are very close approximations to actual figures.)

60 percent of the accidents happened in the school buildings.
 30 percent of the accidents happened on the school playground.
 10 percent of the accidents happened on the way to and from school.

<u>Percent</u>	<u>Number of degrees</u>
60	216° (60% of $360^{\circ} = 216^{\circ}$)
30	
10	

2. In 1956 the figures for the accidents of Problem 1 were as follows: (Again, figures are very close approximations.)
 36 percent of the accidents happened in the school buildings.
 54 percent of the accidents happened on school playgrounds.
 10 percent of the accidents happened on the way to and from school.

3. Tom had an allowance of \$5.00 per week. He spent \$1.75 on lunches, \$1.50 on entertainment, \$1.00 on carfare, \$0.50 on school supplies, and \$0.25 on miscellaneous items.

<u>Items</u>	<u>Amount Spent</u>	<u>Fractional Part of Total</u>	<u>Percent</u>	<u>Number of degrees</u>
Lunch	\$1.75			
Entertainment	1.50			
Carfare	1.00	$\frac{100}{500} = \frac{1}{5}$	20	72
School Supplies	0.50			
Miscellaneous	0.25			
Totals	\$5.00	1	100	360

15-6. Averages.

Some facts can be found easily by looking at all the data in table form. Sometimes, however, a large number of items in a table makes it confusing to understand. Often it is better to describe the collection of data by using the average. Finding an average of such a group of numbers is often very helpful in studying the data given to you.

Do you know how to find an average? You have been doing this for some time, but did you know there are several kinds of "averages"?

Arithmetic Mean

When you calculate the average from a group of numerical grades in spelling by adding the numerical grades and dividing the sum by the number of grades, you accept a single number to represent the whole group of grades. This useful average with which you are already familiar is called the arithmetic mean or the mean. (When the word mean is used in this chapter it always refers to the arithmetic mean.) For example, suppose your 5 spelling grades are 100, 92, 85, 83, and 60. Add them. You get 420. Since there are 5 grades, you divide by 5. The arithmetic mean = $\frac{420}{5}$ or 84. This kind of average is probably familiar to you.

Let us look once more at the heights recorded in Table 15-1a. This table gives a list of the heights, ordered from the tallest to the shortest, of 15 boys.

Table 15-6a

Heights of 15 Seventh Grade Boys

<u>Boys</u>	<u>Heights in Inches</u>
1	65
2	64
3	63
4	61
5	61
6	61
7	60
8	59
9	57
10	55
11	55
12	54
13	54
14	53
15	<u>52</u>
Total	874

One number which describes these data would be the average called the arithmetic mean. In this table the average height (arithmetic mean) is $\frac{874 \text{ (sum of the heights)}}{15} \approx 58$. This average can be computed without arranging the data in any special way. It is a commonly used measure.

Exercises 15-6a

- Find the average of the following groups of numbers.
 - 84, 78, 67, 59, 74, 75, 81, 74.
 - 2116, 1827, 2143, 1865, 1974, 2045.
- One week a bus driver collected the following number of fares: 208, 135, 180, 211 and 195.
- Glenn bowled once a week. His scores for six consecutive weeks were as follows: 161, 148, 156, 172, 150, 167. What was his average score for the six week period?

4. A real estate development contained five lots in one block. The area of the lots were as follows: 17,317 sq. ft., 18,740 sq. ft., 16,236 sq. ft., 19,052 sq. ft., and 15,976 sq. ft. What is the average size of each lot to the nearest whole number?

Median

Another way of describing a collection of data is finding, if possible, a number so that half of the numbers in the group are greater and half are less than the number found.

The median of a group of numbers is found by first arranging the numbers in order of size. It may be found by counting either from smallest to largest or from largest to the smallest. The median is then the middle number.

In the case of the five spelling grades of 100, 92, 85, 83, and 60, you would arrange them in order.

Grades

100	—	Two grades are greater than 85.
92	—	
85	—	MEDIAN
83	—	
60	—	Two grades are less than 85.

85 is then the median of the above list of numbers.

In the set of heights in Table 15-2a the middle number is 59. This is the median of the set. Half of the numbers are greater than 59 and half are less.

Table 15-6b

Boys	Heights in Inches
1	65
2	64
3	63
4	61
5	61
6	61
7	60
8	59 — MEDIAN
9	57
10	55
11	55
12	54
13	54
14	53
15	52

7 boys are
taller than 59 in.

7 boys are
shorter than 59 in.

The median of a collection of numbers is not always a number in the collection. If there is an even number of items in the collection, the median is taken as the average of the two middle numbers. For example, consider these eight numbers 8, 10, 11, 12, 14, 16, 17, 19.

8	
10	
11	
12	
14	> The average or mean of these two "middle" numbers
16	is 13. The median for the whole set is
17	then, 13.
19	

Consider another set of numbers this time with 6 members in it: 12, 13, 15, 15, 17, 20.

22	
13	
15	
15	> The average or mean of these two "middle" numbers
17	(since they are both the same) is 15. The
20	median of this set is 15.

Mode

Which height occurs more than any other in Table 15-2a? How many pupils have this height? This height is called the mode.

In sets which you have studied such as the set of natural numbers, {1, 2, 3, 4, 5, . . .}, no number occurred in the set more than once. But in a set of data some number or numbers may occur more than once. If a number occurs in the set of data more often than any other number it is called the mode. (We might say it is the most fashionable.) There may be several modes. In Table 15-2a there was just one. The number 61 occurs three times. But in the set of scores 19, 20, 21, 21, 21, 24, 26, 26, 26, 29, 30 there are two modes 21 and 26. (These are equally fashionable.) If there had been another of 21 in this set of scores, what would the mode have been? In Table 15-2a if the 12th pupil were 55 inches tall how would this affect the mode?

The range in a collection of data is the difference between the largest and smallest numbers in the collection. In the list of scores given above the range is 11.

Let us now take two lists of test scores and find the arithmetic mean, the median, the mode, and the range.

<u>Scores</u>	
98	Range: $98 - 66 = 32$
95	
90	Median: $\frac{90 + 80}{2} = \frac{170}{2} = 85$
80	
75	Mode: There is no mode.
<u>66</u>	
Total 504	Mean: $\frac{504}{6} = 84$

<u>Scores</u>		
100	}	Range: 100 - 60 = 40
100		
99		
97		
84	—	Median: 84
75	}	Mode: 75
75		
75		
60		
Total 765		Mean: $\frac{765}{9} = 85$

Exercises 15-6b

- The following are a list of test scores: 79, 94, 85, 81, 74, 85, 91, 87, 69, 85, 83.
 - Arrange these scores in a table starting with the highest and ending with the lowest.
 - Find the mode.
 - Find the median.
 - Find the range.
 - Find the mean.
- The following annual salaries were received by a group of ten employees: \$4,000, \$6,000, \$12,500, \$5,000, \$7,000, \$5,500, \$4,500, \$5,000, \$6,500, \$5,000.
 - Find the mean of the data.
 - How many salaries are greater than the mean?
 - How many salaries are less than the mean?
 - Does the mean seem to be a fair way to describe the "average" of this data?
 - Find the median of the set of data.
 - Does the median seem to be a fair way to describe the "average" of this data?

3. Following are the temperatures in degrees Fahrenheit at 6 p.m. for a two week period: 47, 68, 58, 80, 42, 43, 68, 74, 43, 46, 48, 76, 48, 50. Find the:
- Mean
 - Median
 - Range
-

15-7. Summary.

The subject matter of statistics deals, in part, with collecting data, putting it in table form, and representing it by graphs. The tabulating and graphing of the data should be done in such ways that the story told by the data can be easily understood. The broken line graphs, bar graphs, and circle graphs are just a few of the kinds of graphs that may be used.

The next time you see graphs or tables in the magazines, newspapers, or your social studies book, look them over carefully. If averages are mentioned, see if you can tell which average is used. Whenever the kind of average used is not stated, you have a right to question whether the average used gives the best representation of all the data.

To help you recall the new terms you have used in working with statistics, they are listed for you.

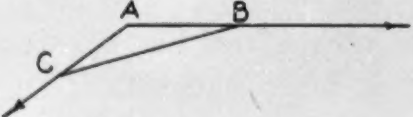
Range--difference between largest and smallest number in a list.

Arithmetic mean or mean--the sum of all the numbers in a list divided by the number of items in the list.

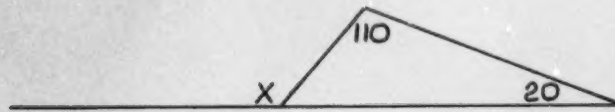
Median--the middle number when data are ordered either from smallest to largest or largest to smallest. When there is no one middle number, the average of the two middle numbers is the median.

Mode--the number occurring most in the list of data. There may be several modes.

15-8. Sample Review Questions.

1. Find the greatest common factor of 8, 12, and 30.
2. The base two numeral for the number twelve has how many digits?
3. What is the intersection of the set of whole numbers between 1 and 11 and the set of even numbers between 5 and 15?
4. Find the simplest numeral for $\frac{16}{25}$ divided by $\frac{8}{5}$.
5. What is the largest number of rays indicated in this diagram?

6. What is the reciprocal for $\frac{2}{9}$ in the set of rational numbers?
7. Explain one way of determining if $\frac{a}{b} = \frac{c}{d}$ is a true statement.
8. If the length of a line segment is measured to the nearest $\frac{1}{2}$ inch, what is the greatest possible error?
9. Find 75% of 14.
10. Find another name for $\frac{7}{8} \cdot \frac{7}{8} \cdot \frac{7}{8} \cdot \frac{7}{8} \cdot \frac{7}{8} \cdot \frac{7}{8}$ using an exponent.
11. If two angles of an isosceles triangle measure 64° and 58° , what is the measure of the third angle?
12. Which of the following measures of central tendency will be one of the original data? Mean, median, mode.
13. Lenore has a picture negative 4 inches wide and 6 inches high. She wants an enlargement that will be 10 inches wide. How high will the enlarged print be?
14. Find a so that the statement is true.
 - a. $\frac{18}{14} = \frac{a}{35}$
 - b. $\frac{a}{100} = \frac{24}{60}$
15. A passenger ship can travel at the rate of 23.8 knots (23.8 nautical miles per hour). How long does it take this ship to travel between two ports that are 345.1 nautical miles apart?

16. What is the measure of $\angle x$ in this figure.



17. Write each of the following in decimal form:
- a. Seven and sixteen thousandths
 - b. Four hundred five and seven hundredths
18. Round each of the following, as indicated.
- a. 57,539 (to the nearest hundred)
 - b. 685,750 (to the nearest thousand)
 - c. 15.0365 (to the nearest hundredth)
 - d. 0.0951 (to the nearest tenth)
-

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